Math 127 Homework

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Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

1. (*) A *partition of* n is a sequence of integers $(a_1, a_2, \dots, a_k)$ such that $a_i \geq 0$ for each i, and $\sum_{i=1}^{k} a_i = n$.

   The number k is called the number of parts of the partition.

   (a) Let $S(n, k)$ denote the set of partitions of n into k parts. Calculate $|S(n, k)|$ for at least 4 different choices of (n, k). Make sure to use both differing n values and differing k values.

   (b) Find a bijection between $\binom{[n+k-1]}{k-1}$ and $S(n, k)$. Conclude that $|S(n, k)| = \binom{n+k-1}{k-1}$.
   Hint: If you're not sure how to construct the bijection, look at some examples with small k (like $k = 2$). It might help to think about the numbers in $[n + k - 1]$ along a number line.

2. Let X and Y be finite sets. Define $X^Y = \{f : Y \to X\}$, the set of all possible functions from Y to X. Prove that $|X^Y| = |X|^{|Y|}$.

3. Prove Theorem 2 in the Infinite Cardinality Notes.

4. (*) Prove Theorem 4 in the Infinite Cardinality Notes. Hint: Use a technique similar to that described in the proof of Theorem 3.

5. Let X be a countable set. Prove that $\{A \subseteq X \mid A \text{ is finite}\}$ is countable.

6. (*) Let $a, b \in \mathbb{R}$ with $a < b$. Prove that $|\{x \in \mathbb{R} \mid a < x < b\}| = |\{x \in \mathbb{R} \mid 0 < x < 1\}|$.

7. Use the technique of Theorem 5 in the Infinite Cardinality Notes to prove Theorem 6.

8. (*) Let $X = \{x \in \mathbb{R} \mid \text{the decimal representation of } x \text{ contains only 4s and 7s}\}$. Is X countable or uncountable? Prove that your answer is correct.

9. (a) Let X be the set of all sequences $(a_1, a_2, a_3, \dots)$ in $\mathbb{N}$ having the property that $a_i \leq a_{i+1}$ for all $i \geq 1$. For example, here is an element of X: $(1, 4, 6, 10, 26, 1583, \dots)$. Is X countable or uncountable? Prove that your answer is correct.

   (b) Let X be the set of all sequences $(a_1, a_2, a_3, \dots)$ in $\mathbb{N}$ having the property that $a_i \geq a_{i+1}$ for all $i \geq 1$. For example, here is an element of X: $(19, 16, 2, 1, 1, 1, 1, \dots)$. Is X countable or uncountable? Prove that your answer is correct.

10. Prove that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.

11. (*) For each of the following sets, determine if it is finite, countably infinite, or uncountable. You need not prove your answer is correct, but you should give a reason for your response.

    (a) $\{x \in \mathbb{R} \mid x^2 \in \mathbb{Z}\}$

    (b) For some $n \in \mathbb{N}$, $\{(a_1, a_2, \dots, a_n) \in \mathbb{N}^n \mid a_1 + a_2 + \cdots + a_n < 5000\}$

    (c) $\{S \in \mathcal{P}(\mathbb{N}) \mid \mathbb{N} \backslash S \text{ is finite}\}$

    (d) $\mathbb{R} \backslash S$, where S is a countable set

    (e) $\{x \in \mathbb{R} \mid \text{there is a decimal expansion of } x \text{ with only even digits}\}$