1 a)

$$S(2,2) = \{(0,2),\ (1,1),\ (2,0)\} \qquad |S(2,2)| = 3$$

$$S(3,2) = \{(0,3),\ (1,2),\ (2,1),\ (3,0)\} \qquad |S(3,2)| = 4$$

$$S(3,3) = \{(0,0,3),\ (0,1,2),\ (0,2,1),\ (0,3,0),$$
$$(1,0,2),\ (1,1,1),\ (1,2,0),\ \cancel{|S(3,3)| = 7}$$
$$(2,0,1),\ (2,1,0),\ (3,0,0)\}$$

$\cancel{S(4,2) = \{(0,4}$

$$|S(3,3)| = 10$$

$$S(4,2) = \{(0,4),\ (1,3),\ (2,2),\ (3,1),\ (4,0)\}$$
$$|S(4,2)| = 5$$

$$S(4,3) = \{(0,0,4),\ (0,1,3),\ (0,2,2),\ (0,3,1),\ (0,4,0),$$
$$(1,0,3),\ (1,1,2),\ (1,2,1),\ (1,3,0),\ (2,0,2),$$
$$(2,1,1),\ (2,2,0),\ (3,0,1),\ (3,1,0),\ (4,0,0)\}$$
$$|S(4,3)| = 15.$$

b) ~~Let~~ Define $F : S(n,k) \to \binom{[n+k-1]}{k-1}$ by

$$F(a) = \left\{ \cancel{l+}\ l + \sum_{i=1}^{l} a_i \;\middle|\; l = 1, \dots, k-1 \right\}. \quad \text{where } a = (a_1, \dots, a_k).$$

First, let's check that this is <u>well defined</u>.

Since $\sum_{i=1}^{l} a_i \le n$ and $l \le k-1$, $\left(\sum_{i=1}^{l} a_i + l\right) \le n+k-1$.

Since for $l_1 < l_2$, and ~~since~~ the $a_i \ge 0$

$l_1 + \sum_{i=1}^{l_1} a_i < l_2 + \sum_{i=1}^{l_2} a_i$, $F(a)$ has exactly $k-1$ elements.

Thus, F is a well-defined function. To see that F is a bijection, ~~we construct~~ we will show injectivity and surjectivity. Let $F(a) = F(b)$. ~~then the~~ ~~[illegible]~~ Since the map $\ell = 1, \ldots, k-1 \mapsto \ell + \sum_{i=1}^{\ell} a_i$ is order preserving, ~~[illegible]~~ this means that

$$\ell + \sum_{i=1}^{\ell} a_i = \ell + \sum_{i=1}^{\ell} b_i \quad \forall \ell = 1, \ldots, k-1. \quad (*)$$

~~For $\ell \in$ we see [illegible]~~ ~~[illegible]~~ Claim: $a_i = b_i \quad \forall i \in \{1, \ldots, k-1\}$.

~~For~~ FTSOC, let j be the least st $a_j \neq b_j$.

If $j = 1$, note $1 + a_1 = 1 + b_1$ ($*$ with $\ell = 1$)

$\Rightarrow a_1 = b_1$ $\times$.

If $j > 1$, $j + \sum_{i=1}^{j-1} a_i + a_j = j + \sum_{i=1}^{j-1} b_i + b_j \Rightarrow a_j = b_j$ $\times$.

Thus $a_i = b_i \quad \forall i \in \{1, \ldots, k-1\}$. But then $a_k = n - \sum_{i=1}^{k-1} a_i$

$$= n - \sum_{i=1}^{k-1} b_i = b_k,$$

(since $a, b \in S(n,k)$)

So in fact $a_i = b_i$ ~~[illegible]~~ $\forall i \in \{1, \ldots, k\}$ and so $a = b$. Thus F is injective.

Now, let ~~[illegible]~~ $S \in \binom{[n+k-1]}{k-1}$ and order S by $\{s_1, s_2, \ldots, s_{k-1}\}$.

Define

$$a_1 = s_1 - 1$$

$$a_i = s_i - s_{i-1} - 1, \quad i \in \{2, \ldots, k-1\}$$

$$a_k = n - \sum_{i=1}^{k-1} a_i.$$

Claim: $(a_1, \ldots, a_k) \in S(n,k)$.

Clearly $a_1 \geq 0$, and since $s_i \geq s_{i-1}+1$, we have $a_i \geq 0$ for $i = 2, \ldots, k-1$,

and $a_k = n - \left(\sum_{i=1}^{k-1} s_i \right) = n - \left(s_1 - 1 + \sum_{i=2}^{k-1} (s_i - s_{i-1} - 1) \right)$

$= n - \left(s_1 + (s_{k-1} - s_1) - (k-1) \right) = n - s_{k-1} + k - 1 \geq 0$

since $s_{k-1} \leq n+k-1$. Thus $a_i \geq 0 \ \forall i \in \{1, \ldots, k\}$

and clearly $\sum_{i=1}^{k} a_i = n$, so $a \in S(n,k)$.

Claim: $F(a) = S$. The elements of $F(a)$ are ordered so it suffices to show that

$F(a)_\ell = \ell + \sum_{i=1}^{\ell} a_i$, so

$\ell + \sum_{i=1}^{\ell} a_i = s_\ell \quad \forall \ell \in \{1, \ldots, k-1\}$. Note

$\ell + \sum_{i=1}^{\ell} a_i = \ell + (s_1 - 1) + \sum_{i=2}^{\ell} \left[\underbrace{(\quad)}_{s_i - s_{i-1}} - 1 \right]$

$= \ell + s_1 + (s_\ell - s_1) - \ell = s_\ell$ ✓.

We conclude that F is a bijection and that $|S(n,k)| = \binom{n+k-1}{k-1}$.

~~2) Let $n = |X|$ and $m = |Y|$. Note that to specify a function $f: X \to Y$, for each $x \in X$ we have m choices for $f(x)$. Thus there are m^n functions, so $|X^Y| = m^n = |X|^{|Y|}$.~~

2) Let $n = |X|$ and $k = |Y|$. To specify a function $f: Y \to X$, for each of the k $y \in Y$ we have n choices for $f(y)$. Thus there are n^k functions, so $|X^Y| = n^k = |X|^{|Y|}$.

3) $1 \Rightarrow 2$: If X is countable, $\exists f: \mathbb{N} \to X$ where f is a surjection. Since Y is countably infinite, $\exists g: Y \to \mathbb{N}$ a bijection. Then $g \circ f: Y \to X$ is a surjection.

$2 \Rightarrow 3$: Let $f: Y \to X$ be a surjection. Since it is a surjection, it has a right inverse $g: X \to Y$ so $f(g(x)) = x$ $\forall x \in X$. Since g is a right inverse, it is an injection.

$3 \Rightarrow 1$: Note that since $g: X \to Y$ is an injection, we have $|X| = |g(X)|$. Since $g(X) \subseteq Y$, $g(X)$ is either finite or countably infinite. Either way, we see X is countable.

4) For $i \in \mathbb{N}$, enumerate X_i as ~~$\{x^i_1, x^i_2, x$~~

$\{x^i_j \mid j \in \mathbb{N}\} = X_i$. Let $D_n = \{x^i_j \mid i+j = n\}$ for $n \in \mathbb{N}$.

<u>Claim</u>; $|D_n| = n-1$ This is true because to pick $i, j \in \mathbb{N}$ st $i+j = n$, ~~the~~ i can range from 1 to $(n-1)$ and then $j = n-i$ is fixed. In any case, we see D_n is finite.

<u>Claim</u>: $\bigcup_{i=1}^{\infty} X_i \subseteq \bigcup_{n=1}^{\infty} D_n$. Since $x \in X_i \Rightarrow x = x^i_j$ for some $j \in \mathbb{N}$, we have $x \in D_{i+j}$.

But $\bigcup_{n=1}^{\infty} D_n$ is the countable union of <u>finite sets</u>, hence countable, so $\bigcup_{i=1}^{\infty} X_i$ is also countable.

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5) Let $F_n = \{A \subseteq X \mid |A| = n\}$, and let $F = \{A \subseteq X \mid A \text{ is finite}\}$

Claim: $F = \bigcup_{n=0}^{\infty} F_n$.

Let $A \in F$. Since A is finite, $|A| \in \mathbb{N}$ (or $|A| = 0$), so $A \in F_{|A|}$.

Thus F is the countable union of ~~finite~~ countable sets, (*) hence countable.

(*) To see F_n countable, note there is a surjection from F_n to $\{f : [n] \to X\}$ and $|\{f : [n] \to X\}|$ is countable.

⑥ ~~Claim~~ Let $I_1 = \{0 < x < 1\}$, $I_2 = \{a < x < b\}$.

Claim: $f: I_1 \to I_2$ $\quad f(x) = (b-a)x + a$

is a bijection.

① f is well defined: $\quad 0 < x < 1 \Rightarrow$

$$0 < (b-a)x < b-a$$

$$\Rightarrow a < \underset{+a}{(b-a)x} < b$$

$$\Rightarrow a < f(x) < b \checkmark.$$

② f ~~is a surjection~~ has an inverse:

Let $g: I_2 \to I_1$ be $g(y) = \dfrac{y-a}{b-a}$.

well defined:

$$a < y < b \Rightarrow 0 < y-a < b-a \Rightarrow 0 < \frac{y-a}{b-a} < 1 \quad \checkmark$$

left inverse: $g(f(x)) = \dfrac{f(x)-a}{b-a} = \dfrac{(b-a)x + a - a}{b-a} = x \quad \checkmark$

right inverse: $f(g(y)) = (b-a)g(y) + a = (b-a)\left(\dfrac{y-a}{b-a}\right) + a = y \quad \checkmark$

Thus $\exists$ bijection from I_1 to I_2, so $|I_1| = |I_2|$.

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⑦ Assume, for the sake of contradiction, that $\exists f: X \to P(X)$ surjective. Let $D := \{x \in X : x \notin f(x)\} \subseteq X$. Since $D \subseteq X$, $D \in P(X)$, so by surjectivity, $\exists x_0 \in X$ st $f(x_0) = D$.

Now, if $x_0 \in D$, then $x_0 \in f(x_0)$, so $x_0 \notin D$. ⨳

On the other hand, if $x_0 \notin D$, then $x_0 \in f(x_0)$, so $x_0 \in D$ ⨳.

Since either way we have a contradiction, we conclude there is <u>no</u> surjection from X to $P(X)$.

(8) Let $f: \{4,7\} \to \{4,7\}$ be defined as $f(4)=7$, $f(7)=4$.

FTSOC, say we can enumerate X as $\{x_i \mid i \in \mathbb{N}\}$ and let $x_{i,j}$ be the j^{th} digit of x_i for all $j \in \mathbb{N}$. Define ~~b~~ $b \in \mathbb{R}$ so that the j^{th} digit of b is $f(x_{j,j})$ for $j \in \mathbb{N}$. Since $f(x_{j,j}) \in \{4,7\}$, the digits of b are 4 and 7 so $b \in X$. However, I claim $b \neq x_i$ $\forall i \in \mathbb{N}$. Indeed, since the i^{th} digit of b is $f(x_{i,i}) \neq x_{i,i}$, we have $b \neq x_i$. This contradicts our claim that $\{x_i \mid i \in \mathbb{N}\} = X$, so we ~~have a contradi~~ conclude that X is uncountable.

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9) a) X is uncountable. Let ~~B~~ Y be the set of all sequences in $\mathbb{N} \cup \{0\}$, which is clearly uncountable. The map $f: X \to Y$, $f(a)_i = a_{i+1} - a_i$, $i \in \mathbb{N}$ is a bijection.

~~b) X is countable. Note that the sequences which become 1 after some finite~~

~~10) It is enough to ...~~

11)

a) Countable infinite. ~~$x^2 \in \mathbb{Z}$~~ $x^2 \in \mathbb{Z} \Rightarrow x = \sqrt{y}$ or $-\sqrt{y}$ for $y \in \mathbb{Z}$, and there are countably many $y \in \mathbb{Z}$.

b) Finite: each a_i is between 1 and 5000.

c) Countably infinite: The finite subsets are countably infinite, and $S \mapsto \mathbb{N} \setminus S$ is a bijection.

d) Uncountable: If not, $\mathbb{R} = (\mathbb{R} \setminus S) \cup S$ the union of countable sets.

e) Uncountable: ~~Skew~~ we could diagonalize by sending the digits ~~the~~ 2-8 to 0 and the digit 0 to 2.