1. Claim 1: Given $g: [n+1] \to X$ bijective

$g(n+1) \notin S$, $X' = X \setminus \{g(n+1)\}$

$h: [n] \to X'$ $h(k) = g(k)$ a bijection so $\exists h': [n] \to X'$ bijection, having $(h')^{-1}(S) = [k]$.

Claim: $f: [n+1] \to X$ $f(m) := \begin{cases} h'(m) & 1 \le m \le n \\ g(n+1) & m = n+1 \end{cases}$ is a bijection.

<u>Injective:</u> Let $i, j \in [n+1]$, $i \neq j$. If $i \in [n]$ and $j \in [n]$, then $f(i) = h'(i)$, $f(j) = h'(j)$, and h' injective $\Rightarrow h'(i) \neq h'(j)$.

If one of $i, j = n+1$ (WLOG, $i = n+1$) then $j \neq n+1$.

$f(i) = g(n+1)$, $f(j) = h'(j) \in X'$, Since $g(n+1) \notin X'$, $f(i) \neq f(j)$.

<u>Surjective:</u> $f([n+1]) = f([n]) \cup f(\{n+1\}) = h'([n]) \cup f(\{n+1\})$

$= X' \cup \{g(n+1)\} = X$.

Claim 2: When $g(n+1) \in S$, $X' = X \setminus \{g(n+1)\}$, $S' = S \setminus \{g(n+1)\}$

and $\exists$ bij $h': [n] \to X'$, $(h')^{-1}(S') = [k]$.

$f: [n+1] \to X$ $f(m) = \begin{cases} g(n+1) & m = 1 \\ h'(m-1) & 2 \le m \le n+1 \end{cases}$

is a bijection.

<u>Inj:</u> Identical to proof of <u>Claim 1</u> if $i, j \in \{2, \dots n+1\}$.

If $i = 1$, $j \neq 1$ so $f(j) = h'(j-1) \in X'$ and

$f(i) = g(n+1) \notin X' \Rightarrow f(i) \neq f(j)$.

<u>Surj:</u> $f([n+1]) = \{g(n+1)\} \cup h'([n]) = \{g(n+1)\} \cup X' = X$.

(2) WLOG, let X be finite. Since $X \cap Y \subseteq X$, by Corollary 1 we have $X \cap Y$ finite as well.

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(3) We will find a bijection $F: S \to \mathcal{P}(X) \setminus S$.

~~For $A \in S$, $F(S) := X \setminus A$. Note~~

Fix some $x_0 \in X$. $F(A) := \begin{cases} A \cup \{x_0\} & \text{if } x_0 \notin A \\ A \setminus \{x_0\} & \text{if } x_0 \in A. \end{cases}$

Since F either adds or subtracts 1 to $|A|$, we see $F: S \to \mathcal{P}(X) \setminus S$ is well defined.

To see that F is a bijection, note that $G: \mathcal{P}(X) \setminus S \to S$

$$G(B) = \begin{cases} B \cup \{x_0\} & \text{if } x_0 \notin B \\ B \setminus \{x_0\} & \text{if } x_0 \in B \end{cases}$$

is an inverse function, since we either add and subtract x_0 or subtract x_0 and add it back.

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(4) If $S \subseteq X$, $S \neq X$, then $X \setminus S \neq \emptyset$, so $|X \setminus S| > 0$.

Thus $|X| = |S| + |X \setminus S| > |S|$.

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(5) Proof by counting: To choose an element of S, we select 1 element from B and then any subset of $A \setminus B$. Thus,

$$|S| = k \cdot 2^{n-k}.$$

⑥ Proof by induction. Claim: $\forall n, \forall x,y \in \mathbb{R}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Base case: $n=1$ $\quad (x+y) = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0$ ✓

If true for n, then

$$(x+y)^{n+1} = (x+y) \cdot \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(x^{k+1} y^{n-k} + x^k y^{n+1-k}\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} x^k y^{n+1-k} \cdot \binom{n}{k}$$

$$= \sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=0}^{n} x^k y^{n+1-k} \binom{n}{k}$$

$$= y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} + x^{n+1}$$

$$= y^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] x^k y^{n+1-k} + x^{n+1}$$

↓ Pascal's Identity

$$= y^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + x^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$ ✓

(7) Apply the binomial theorem to $x=1$, $y=-1$.

~~Then~~ $\cancel{(-1)^n = (0-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k}$

Then $$0^n = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

so $$1 + \sum_{k=1}^{n} \binom{n}{k} (-1)^k = 0 \iff \sum_{k=1}^{n} \binom{n}{k} (-1)^k = -1.$$

(8) Let S_F = permutations of X that contain "fish"

S_C = " " " "cat"

S_M = " " " "mouse".

S = all permutations of X.

$$|S \setminus (S_F \cup S_C \cup S_M)| = |S| - |S_F \cup S_C \cup S_M|$$

$$= |S| - |S_F| - |S_C| - |S_M| + |S_F \cap S_C| + |S_F \cap S_M| + |S_C \cap S_M| - |S_F \cap S_C \cap S_M|.$$

by inclusion/exclusion thm.

Now, $|S| = 26!$, the # of permutations.

For S_F, S_C, S_M, we view the word as a "block" of text and all the remaining characters as individual blocks, so there are $(26-\ell)+1 = 27-\ell$ blocks to arrange, and thus

(ℓ = length of word)

$|S_F| = (27-4)! = 23!$

$|S_C| = (27-3)! = 24!$

$|S_M| = (27-5)! = 22!$

For $S_F \cap S_C$, we consider <u>both</u> words as individual blocks, and the remaining characters as individual blocks, so if the words have length ℓ_1 and ℓ_2, there are $((26-\ell_1-\ell_2)+2)!$ ways to arrange the blocks, ie

$$|S_F \cap S_C| = (21)!, \quad |S_C \cap S_M| = (20)!$$

Note that $S_M \cap S_F = \emptyset$ since "s" cannot precede both "h" and "e", so $|S_M \cap S_F| = |S_F \cap S_C \cap S_M| = 0$.

Thus

$$|S \setminus (S_F \cup S_C \cup S_M)| = 26! - 23! - 24! - 22! + 21! + 20!$$

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⑨ Let $S_n = \{ k \in [1000] \mid n \mid k \}$.

Note $|S_n| = \left\lfloor \frac{1000}{n} \right\rfloor$, since every n integers, we get 1 more multiple of n.

Now, $|[1000] \setminus (S_5 \cup S_7 \cup S_{12})| =$

$\hookrightarrow$ (I.E.) $1000 - |S_5| - |S_7| - |S_{12}| + |S_5 \cap S_7| + |S_7 \cap S_{12}| + |S_5 \cap S_{12}|$
$- |S_5 \cap S_7 \cap S_{12}|$.

Note $S_5 \cap S_7 = S_{35}$, since 5, 7 have no common factors, so $5 \mid x$ and $7 \mid x \leftrightarrow 35 \mid x$.

Similarly, $S_7 \cap S_{12} = S_{84}$ $\quad S_5 \cap S_7 \cap S_{12} = S_{420}$

$S_5 \cap S_{12} = S_{60}$

$$|[1000] \setminus (S_5 \cup S_7 \cup S_{12})| = 1000 - \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{1000}{12} \right\rfloor$$

$$+ \left\lfloor \frac{1000}{35} \right\rfloor + \left\lfloor \frac{1000}{84} \right\rfloor + \left\lfloor \frac{1000}{60} \right\rfloor - \left\lfloor \frac{1000}{420} \right\rfloor$$

$$= 1000 - 200 - 142 - 83 + 28 + 11 + 16 - 2 = 628.$$

(10) We can count this by picking the 4 games in which team A wins. ~~Every win~~ Let the results be written in a string, like ABAABA ← A wins!
1 2 3 4 5 6

~~we can~~ Since the string ends when A has exactly 4 wins, we can pad the end with Bs to make the string length 7.
ABAABA : B. This string always has 7 characters, 4 As.
there are $\binom{7}{4} = 35$ ways.

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(11) a) There are $(n-1)!$ ways to get my own coat back, since ~~there~~ we pick where the 1st other coat goes and there are $(n-1)$ slots, &c.

b) For me and Tim, there are $(n-2)!$ ways, since 2 slots are already fixed.

c) There are $(n-k)! \times k!$ ways that our house goes back with the right coats in any order

d) There are $(n-k)!$ ways that our house gets our coats back in the right order.

(12) Prove that $\forall n$, $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Counting 2 ways: Let N be the # of ways to split $2n$ people into team A & team B of equal size.

Clearly $\binom{2n}{n} = N$, just pick n people to be in team A.

Or, we note $\displaystyle\sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$.

Now, if n people have brown hair and n people have blonde hair, we can pick teams by first choosing k brown haired people for Team A & the other $(n-k)$ Team A with blonde hair.

k may range from $k=0$ to n.

Thus $\displaystyle N = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$.

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(13) Prove that

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m}$$

Counting 2 ways: Let N be the # of ways to choose a club of n people that has a leadership board of size m.

$N = 2^{n-m}\binom{n}{m}$, since we can pick the board $\binom{n}{m}$ ways and then pick the other members from the remaining $(n-m)$ people 2^{n-m} ways.

but $N = \sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}$

Since we can pick the club first and then choose m members to promote to the board. The club could have anywhere from m to n members, and if it has size k, there are $\binom{n}{k}$ waysto pick the club and $\binom{k}{m}$ ways to pick the board. Thus

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m}.$$