(1a) "$\Rightarrow$" f is injective $\Rightarrow \exists$ left inverse $g: Y \to X$

st $g(f(x)) = x \quad \forall x \in X$.

Claim: g is surjective. For $x \in X$, note $f(x) \in Y$

and $g(f(x)) = x$. $\blacksquare$

"$\Leftarrow$" g is surjective $\Rightarrow \exists$ right inverse $f: X \to Y$.

st $g(f(x)) = x \quad \forall x \in X$

Claim: f is injective.

~~$\forall x_1 \neq x_2$~~ If $f(x_1) = f(x_2)$, then

$g(f(x_1)) = g(f(x_2)) \Rightarrow x_1 = x_2$. $\blacksquare$

(1b) WLOG, we can just show $\Rightarrow$. Since $f: X \to Y$

is a bijection, $\exists g: Y \to X$ inverse. By 1a,

g is surjective & injective $\Rightarrow$ g is bijective.

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(3) Let $X = Y = \mathbb{N}$, $f(n) = n+1$. Since $n+1 = m+1 \Rightarrow n = m$,

f is injective, but since $f(n) \geq 2 \ \forall n$, f is not surjective.

However, there is a trivial bijection from $\mathbb{N}$ to $\mathbb{N}$, the identity.

The claim is true if X is <u>finite</u>. PF.

Case 1: Y is infinite. Then, there cannot be a bijection from X to Y, so it is vacuously true.

Case 2: Y is finite. Since $\exists f: X \to Y$ ~~be~~ injective but not surjective, $Y \setminus f(X) \neq \emptyset$ and $|Y \setminus f(X)| > 0$.

But $g: X \to f(X)$, $g(x) = f(x)$ <u>is</u> a bijection, so

$$|f(X)| = |X| \text{ and thus } |Y| = |f(X)| + |Y \setminus f(X)|$$
$$> |X| + 0.$$

Since $|X| < |Y|$, there cannot be a bijection.

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(4) Note f inj & g ~~inj~~ inj $\Rightarrow$ $g \circ f$ inj

f surj & g surj $\Rightarrow$ $g \circ f$ surj

$g \circ f$ inj & $g \circ f$ surj $\Rightarrow$ $g \circ f$ <u>bijection</u>.

Now, want to show $(g \circ f)^{-1}(z) = f^{-1}(g^{-1}(z))$ $\forall z \in Z$.

Let $x = (g \circ f)^{-1}(z)$. Then $g(f(x)) = z$, so

$$f^{-1}(g^{-1}(z)) = f^{-1}(g^{-1}(g(f(x)))) = f^{-1}(f(x)) = x$$
$$= (g \circ f)^{-1}(z).$$

since $g^{-1}(g(f(x))) = f(x)$

∎

(6) Let $i(x) = \{x\}$. Clearly i is injective, since $\{x_1\} = \{x_2\} \Rightarrow x_1 = x_2$. But i is not surjective: $\emptyset \in P(X)$, but $|\emptyset| = 0$, and $|i(x)| = 1 \ \forall x \in X$. By (3), since X is finite, there cannot exist a bijection from $X \to P(X)$ since we have an injection that fails to be surjective, so $|X| < |P(X)|$.

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(9) "$\Rightarrow$" Note that f injective $\Rightarrow$ $g : X \to f(X)$, $g(x) = f(x)$ is a bijection. For $A \subseteq Y$,

$$|A| = |A \cap f(X)| + |A \setminus f(X)| \geq |A \cap f(X)| = |g^{-1}(A)| = |f^{-1}(A)|$$

~~since~~

"$\Leftarrow$" Contrapositive: say $\exists A \subseteq Y$, $|f^{-1}(A)| > |A|$. Note: $\forall x \in f^{-1}(A)$, $f(x) \in A$. By pigeonhole principle, there must be $x_1 \neq x_2 \in f^{-1}(A)$ st $f(x_1) = f(x_2)$. ■