21-127 HW6 SOL'NS

① a) i) Claim: $A_p \cap A_q \subseteq A_{pq}$

Let $x \in A_p \cap A_q$. Then $p \mid x$ and $q \mid x$, so the prime factorization of x contains p & q. Thus, $x = pq \cdot \prod_{i=1}^{M} r_i$ where the r_i are the other prime factors of x, so $x \in A_{pq}$. Thus $A_p \cap A_q \subseteq A_{pq}$.

ii) Claim: $A_{pq} \subseteq A_p \cap A_q$. Let $x \in A_{pq}$. Then $\frac{x}{pq} \in \mathbb{Z}$, so $p \cdot \frac{x}{pq} = \frac{x}{q} \in \mathbb{Z}$ and $q \cdot \frac{x}{pq} = \frac{x}{p} \in \mathbb{Z}$, so $x \in A_q$ and $x \in A_p$, ie $x \in A_p \cap A_q$. Thus $A_{pq} \subseteq A_p \cap A_q$.

b) If $a \mid b$, WTS $A_b \subseteq A_a$. Let $x \in A_b$, so $b \mid x$. But $a \mid b$ and $b \mid x \Rightarrow a \mid x$, so $x \in A_a$. Thus $A_b \subseteq A_a$.

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② a) i) WTS $X \subseteq Y \to X \cap Y = X$.

Let $X \subseteq Y$. Since $X \cap Y \subseteq X$, it suffices to show $X \subseteq X \cap Y$. Let $x \in X$. Since $X \subseteq Y$, we have $x \in Y$. Thus $x \in X \to (x \in X \text{ and } x \in Y)$, so $X \subseteq X \cap Y$.

ii) WTS $X \cap Y = X \to X \subseteq Y$.

Let $X \cap Y = X$. Fix $x \in X$. Since $x \in X$, $x \in X \cap Y$ as well by $X = X \cap Y$. But $x \in X \cap Y \Rightarrow x \in Y$, so $X \subseteq Y$.

b) ~~Consider $A = X^c$, $B = Y^c$. Apply (2a) to A and B to see~~

Apply (2a) to Y^c and X^c to see

$Y^c \subseteq X^c \leftrightarrow (Y^c \cap X^c) = Y^c$.

But $Y^c \subseteq X^c \leftrightarrow X \subseteq Y$, and by De Morgans law,

$(Y^c \cap X^c) = Y^c \Leftrightarrow Y \cup X = Y$.

Thus $X \subseteq Y \Leftrightarrow X \cup Y = Y$.

~~3a) Let $x \in ((\bigcup_{i=1} A_i)^c$. Then~~

(5) Let A, B be given.

Claim: $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$

and that $A \setminus B$, $B \setminus A$, $A \cap B$ are pairwise disjoint.

The key observation is that $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$ requires at least one of $x \in A$, $x \in B$ to be true, which can be satisfied in exactly one of the following ways:

$x \in A$ and $x \notin B \leftrightarrow x \in A \setminus B$

$x \notin A$ and $x \in B \leftrightarrow x \in B \setminus A$

$x \in A$ and $x \in B \leftrightarrow x \in A \cap B$.

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(7) The Archimedean Property: $\forall a, b \in \mathbb{N}$, $\exists n \in \mathbb{N}$ st $na > b$.

Let $a \in \mathbb{N}$ be arbitrary, and let $S = \{b \in \mathbb{N} \mid n \cdot a \leq b \ \forall n \in \mathbb{N}\}$

FTSOC, say that $S \neq \emptyset$, so by the well-ordering principle it has a minimal element $b_0 \in S$. Since $b_0 \in S$, $2a \leq b_0$ and thus $b_0 - a > 0$, ie $b_0 - a \in \mathbb{N}$. But then since $\forall n \in \mathbb{N}$ $(n+1)a \leq b_0$, we have $n \cdot a \leq b_0 - a$, so $b_0 - a \in S$. This contradicts the minimality of b_0 and we conclude $S = \emptyset$, ie $\forall b \in \mathbb{N}$ $\exists n \in \mathbb{N}$ st

(12) a) Let $U \subseteq X$, WTS $f(X \setminus U) \supseteq f(X) \setminus f(U)$.

Let $y \in f(X) \setminus f(U)$, so $\exists x \in X$ st $f(x) = y$, but $y \neq f(u)$ for all $u \in U$. Thus $x \in X \setminus U$, since if $x \in U$, we would have $f(x) \neq y$, but we know $f(x) = y$. Since $x \in X \setminus U$ and $y = f(x)$, $y \in f(X \setminus U)$.

b) ~~Let $x \in f^{-1}$ i) WTS $f^{-1}(Y \setminus V) \subseteq X \setminus f^{-1}(V)$. Let $x \in f^{-1}(Y \setminus V)$. Then $f(x) \in Y \setminus V$, ie $f(x) \notin V$~~

Let $x \in X$ be fixed. Then

$x \in f^{-1}(Y \setminus V) \Leftrightarrow f(x) \in Y \setminus V \Leftrightarrow f(x) \notin V \Leftrightarrow x \notin f^{-1}(V) \Leftrightarrow x \in X \setminus f^{-1}(V)$.

Thus $X \setminus f^{-1}(V) = f^{-1}(Y \setminus V)$.

(15) a) Injective: No, $f(1) = 1$ and $f(-1) = 1$.
Surjective: No, $2 \in \mathbb{N} \cup \{0\}$ but $2 \neq x^2$ $\forall x \in \mathbb{Z}$ since $\sqrt{2}$ is irrational.

b) Injective: Yes, if $x^2 = y^2$ then $x = y$ or $x = -y$ $\forall x, y \in \mathbb{Z}$, but since $x, y \in \mathbb{N}$ we have $x = y$.
Surjective: No, $-1 \in \mathbb{Z}$ but $x^2 > 0$ $\forall x \in \mathbb{N}$.

c) Injective: No, $f(1) = 1$ and $f(1.5) = 1$.
Surjective: Yes, $h(k) = k$ $\forall k \in \mathbb{Z}$ and $\mathbb{Z} \subseteq \mathbb{R}$.

d) Injective: Yes. ~~Note that even numbers map to and odd numbers map to negatives, so it is enough to prove injectivity for x, y even or x, y both odd.~~

If x, y both even and $\frac{x}{2} = \frac{y}{2}$, then $x = y$.

If x, y both odd and $\frac{-(x-1)}{2} = \frac{-(y-1)}{2}$, then $x-1 = y-1$ and $x = y$.

~~Thus we have injectivity.~~

If x is even and y is odd, note $\frac{x}{2} \geq 1$ since $x \geq 2$ while $y \geq 1 \Rightarrow y-1 \geq 0 \Rightarrow \frac{-(y-1)}{2} \leq 0$. Thus

$x \neq y \Rightarrow f(x) \neq f(y)$ and by contrapositive

$f(x) = f(y) \Rightarrow x = y$. Thus we have injectivity.

Surjectivity: Yes. Let $k \in \mathbb{Z}$ be arbitrary.

If $k \geq 1$, then $2k \in \mathbb{N}$ and $2k$ is even, so

$f(2k) = \frac{2k}{2} = k$.

If $k \leq 0$, then $1-2k \geq 1$, ie $1-2k \in \mathbb{N}$ and $1-2k$ is odd, so

$f(1-2k) = -\frac{(1-2k-1)}{2} = k$. Thus f is surjective.

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(17) Note $g \circ f$ surjective $\Leftrightarrow g(f(X)) = Z$.

Then since $g(Y) \subseteq Z$, suffices to show $Z \subseteq g(Y)$.

But $f(X) \subseteq Y \Rightarrow g(f(X)) \subseteq g(Y) \Rightarrow Z \subseteq g(Y)$.