HW5 SOL'NS

① Induction: $P(n)$: " $2^n < n!$ "

$P(4)$: $2^4 = 16 < 24 = 4!$ Base Case ✓

Inductive Step: Claim $\forall n \geq 4$, $P(n) \to P(n+1)$.

$$2^{n+1} = \underbrace{2^n \cdot 2 < n! \cdot 2}_{\text{By } P(n)} \underbrace{<}_{\text{since } n>1} n! \cdot (n+1) = (n+1)! \checkmark$$

② $P(n)$: $\sum_{k=1}^{n} k!k = (n+1)! - 1$

$P(1)$: $1! \cdot 1 = 1 = 2! - 1$ ✓ Base Case

Inductive step: Assume $P(n)$. Then

$$\sum_{k=1}^{n+1} k!k = \sum_{k=1}^{n} k!k + (n+1)!(n+1)$$

$$= (n+1)! - 1 + (n+1)!(n+1) = (n+1)!(1+n+1) - 1$$

$$= (n+1)!(n+2) - 1 = (n+2)! - 1$$ so $P(n+1)$ holds.

③ Base case: $\prod_{i=1}^{1} (2i-1) = 1 = \frac{(2)!}{2 \cdot 1!}$

Inductive step: $\prod_{i=1}^{n+1} (2i-1) = \prod_{i=1}^{n} (2i-1) \cdot (2 \cdot (n+1) - 1)$ *

$$= \frac{(2n)!}{2^n n!} \times (2n+1) = \frac{(2n+2)! \times \frac{1}{(2n+2)}}{2^n \cdot n!} = \frac{(2n+2)!}{2 \cdot 2^n \cdot n! \cdot (n+1)} = \frac{(2(n+1))!}{2^{n+1}(n+1)!} \checkmark$$

④ Strong Induction. $P(n)$: $a_n = 4^n - 3^n$.

Base case: $n = 0, 1$.

$4^0 - 3^0 = 0 = a_0$ ✓

$4^1 - 3^1 = 1 = a_1$ ✓

Claim: $\forall n \geq 1$, if $P(n)$ $\forall k \leq n$, we have $P(n+1)$.

PF: Since $n+1 \geq 2$,

$a_{n+1} = 7a_n - 12a_{n-1}$. Since $P(n)$, $P(n-1)$ hold,

$$a_{n+1} = 7\cdot(4^n - 3^n) - 12(4^{n-1} - 3^{n-1})$$
$$= 7\cdot 4^n - 3\cdot 4^n - 7\cdot 3^n + 4\cdot 3^n$$
$$= 4\cdot 4^n - 3\cdot 3^n = 4^{n+1} - 3^{n+1} \checkmark$$

Thus $P(n+1)$.

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⑤ Since we prove for n odd, we need to prove $P(n) \rightarrow P(n+2)$ with a base case of $P(1)$.

Base case: $1^2 - 1 = 0$ and $8 \mid 0$ ✓.

Inductive Step: Assume $P(n)$. Then

$$(n+2)^2 - 1 = n^2 + 4n + 4 - 1 = (n^2 - 1) + 4(n+1).$$

Since $8 \mid (n^2-1)$ by $P(n)$, enough to see $8 \mid 4(n+1)$. But n odd $\Rightarrow$ $n+1$ even $\Rightarrow$ $n+1 = 2k$ for some $k \in \mathbb{Z}$, so $4(n+1) = 8k$ is divisible by 8.

6 ~~Case~~ Case 1: In the n-game, where there are 2 even stacks of n pennies, player 2 has a winning strategy.

$P(n)$.

Proof by strong induction.

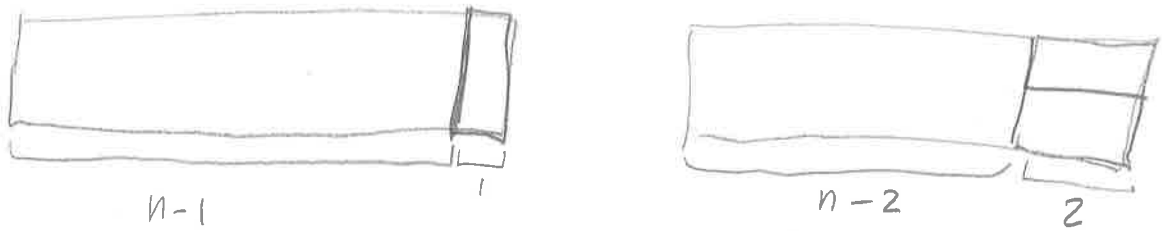
$P(1)$ is trivial. Player 1 must eliminate one stack, so Player 2 removes the other.

Now, assume $P(k)$ $\forall k \leq n$. Consider the $(n+1)$-game. ~~Player~~ Player 1 must remove ℓ pennies from one stack. If $\ell = n+1$, Player 2 can win by eliminating the other stack. Else, Player 2 should remove ℓ pennies from the other stack, so now Player 1 has the first move of an $(n+1-\ell)$-game, where Player 2 has a winning strategy by $P(n+1-\ell)$.

Case 2: If the stacks are uneven, Player 1 should remove from the larger stack to make them even. Then Player 2 must make the 1st move of an even-game, so Player 1 is now the "Player 2" of an even-game.

7 Let $n \geq 3$. Claim: $D_n = D_{n-1} + D_{n-2}$.

~~Consider the topmost~~ We count the ways to ~~make~~ tile ~~on~~ a $2 \times n$ board.



the rightmost domino must either be one vertical on 2 horizontal. After making this decision, we see that rightmost-vertical boards could be completed by any $(n-1)$ board. The same is true for rightmost-horizontal boards and $(n-2)$.

Thus $D_n = D_{n-1} + D_{n-2}$.

~~Claim~~ Let φ, ψ be the roots of $x^2 = x + 1$

$\left(\text{Explicitly: } \varphi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2} \right)$

Claim: $D_n = \dfrac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi}$. Proof by strong induction.

$D_1 = 1$ $\qquad \dfrac{\varphi^2 - \psi^2}{\varphi - \psi} = \varphi + \psi = 1$ ✓

$D_2 = 2$

$$\frac{\varphi^3 - \psi^3}{\varphi - \psi} = \varphi^2 + \varphi\psi + \psi^2 = \varphi + 1 + \varphi\psi + \psi + 1$$

$$= 3 + \varphi\psi$$

$$= 3 + \tfrac{1}{4}(1-5) = 2$$ ✓.

Inductive Step: Claim $D_n = \dfrac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi}$ $\forall k \le n$,

WTS $D_{n+1} = \dfrac{\varphi^{n+2} - \psi^{n+2}}{\varphi - \psi}$.

$D_{n+1} = D_n + D_{n-1}$

$$= \frac{1}{\varphi - \psi} \cdot \left(\varphi^{n+1} - \psi^{n+1} + \varphi^n - \psi^n \right)$$

But $\varphi^2 = \varphi + 1 \Rightarrow \varphi^{n+2} = \varphi^{n+1} - \varphi^n$

$\psi^2 = \psi + 1 \Rightarrow \psi^{n+2} = \psi^{n+1} - \psi^n$

$$= \frac{1}{\varphi - \psi} \cdot \left(\varphi^{n+2} - \psi^{n+2} \right).$$

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(8) Clearly the base case $1 = 1 \cdot 1!$ is satisfied.

We claim that if n has a representation $n = \sum_{i=1}^{r} d_i (i!)$, $d_i \le i$, so does $n+1$.

We will use b_i to denote the digits of $n+1$, to be determined. ~~If $d_1 \neq 0$, set~~

If $d_1 = 0$, set $b_1 = 1$ and $b_i = d_i$ $\forall\ 2 \le i \le r$,

so $\sum_{i=1}^{r} b_i \times i! = 1 + \sum_{i=1}^{r} d_i \times i! = 1 + n$.

If $d_1 = 1$, let k be the largest number such that

$d_i = i$ $\forall\ i \le k$.

If $K = r$, then $n = \sum_{i=1}^{r} i \times i! = (r+1)! - 1$

and we can write by Problem 2.

Thus $n+1 = (r+1)! = \sum_{i=1}^{r+1} b_i \times (i)!$

where $b_i = \begin{cases} 0 & i \le r \\ 1 & i = r+1 \end{cases}$.

If $K < r$, then $d_{K+1} < K+1$ by definition of K,

and

$$n = \sum_{i=1}^{K} i \times i! + \sum_{i=K+1}^{r} d_i \times i!$$

$$= (K+1)! - 1 + \sum_{i=K+1}^{r} d_i \times i!$$

again by Problem 2.

Thus $n+1 = (K+1)! + \sum_{i=K+1}^{r} d_i \times i!$

and we can set $b_i = \begin{cases} 0 & i \le K \\ d_i + 1 & i = K+1 \\ d_i & i > K+1 \end{cases}$

where the assignment $b_{K+1} = d_{K+1} + 1$ is valid since $d_{K+1} < K+1$.

Problems 9, 10 excluded due to the presence of Set Theory