HW4 SOL'NS

① If $x = n_0.d_1 d_2 d_3 \ldots$ and

$x_k = n_0.d_1 d_2 \ldots d_k$, then

$$|x - x_k| = |0.000\ldots 0 d_{k+1} d_{k+2} \ldots|$$

$$\leq |0.000\ldots 0999\ldots| = |0.\underbrace{00\ldots01}_{k \text{ places}}| = \frac{1}{10^k}$$

Let $\varepsilon > 0$ be fixed. Let k_0 be st $10^{k_0} > \frac{1}{\varepsilon}$, then

$\forall k > k_0$, $|x - x_k| \leq \frac{1}{10^k} \leq \frac{1}{10^{k_0}} < \varepsilon$.

Thus $\lim_{k \to \infty} x_k = x$.

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② Lemma: $1 \cdot m = m \cdot 1 = m \quad \forall m \in \mathbb{N}$

PF: By induction on m. Base case: $1 \cdot 1 = 1 \cdot 1 = 1$.

Inductive step: Claim $\underbrace{1 \cdot m = m \cdot 1}_{=m}$, want to show $1 \cdot m^+ = \underbrace{m^+ \cdot 1}_{= m^+}$

PF: $\underline{1 \cdot m^+} = 1 + 1 \cdot m = 1 + m$

$\underline{m^+ \cdot 1} = \underline{m^+} = 1 + m$ thus $1 \cdot m^+ = m^+ \cdot 1 = m^+$.

Conclude: $1 \cdot m = m \cdot 1 = m \quad \forall m \in \mathbb{N}$.

Main Result: $n \cdot m = m \cdot n \quad \forall n, m \in \mathbb{N}$

PF: By induction on n. Base case: See Lemma.

Inductive step: Claim $n \cdot m = m \cdot n \ \forall m \in \mathbb{N}$, WTS $n^+ \cdot m = m \cdot n^+ \ \forall m$.

PF: Fix $m \in \mathbb{N}$. Then $n^+ \cdot m = (n+1) \cdot m = n \cdot m + m$

and $m \cdot n^+ = m + m \cdot n = m \cdot n + m \underset{\text{By inductive hyp.}}{=} n \cdot m + m$

Thus $n^+ \cdot m = m \cdot n^+$.

③ WTS $\forall a,b,c \in \mathbb{N}$, $(a+b)+c = a+(b+c)$.

Pf by induction on c.

Base case: $(a+b)+1 = (a+b)^+ = a+b^+$

and $a+(b+1) = a+b^+$.

Thus $(a+b)+1 = a+(b+1)$.

Inductive step: Show that if $(a+b)+c = a+(b+c)$

then $(a+b)+c^+ = a+(b+c^+)$.

$$(a+b)+c^+ = ((a+b)+c)^+$$

$$a+(b+c^+) = a+(b+c)^+ = (a+(b+c))^+$$

Since $(a+b)+c = a+(b+c)$ by Inductive Hypothesis

conclude $(a+b)+c^+ = a+(b+c^+)$.

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④ Base Case: $n=1$, $4^1+6\cdot 1-1 = 9$ is divisible by 9.

Inductive step: Let $4^n+6n-1 = 9\cdot \ell$ for some $\ell \in \mathbb{Z}$.

Then $4^n = 9\cdot\ell - 6n+1$ and so

$$4^{n+1}+6(n+1)-1 = 4\cdot 4^n + 6n+6-1$$

$$= 4\cdot(9\ell-6n+1)+6n+5 = 9\cdot 4\ell - 24n+4+6n+5$$

$$= 9\cdot 4\ell - 18n+9 = 9\cdot(4\ell-2n+1).$$

Thus $4^{n+1}+6(n+1)-1$ is divisible by 9.

⑤ Base case: $\sum_{i=1}^{1} \frac{1}{i^2} = 1 \leq 2 - \frac{1}{1}$ ✓

Inductive step: ~~Claim~~ If $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$,

WTS $\sum_{i=1}^{n+1} \frac{1}{i^2} \leq 2 - \frac{1}{n+1}$. ~~Note that since $n \leq n+1$,~~

~~$n(n+1) \leq (n+1)^2$~~ Note that since $n^2+2n \leq n^2+2n+1$,

$\Rightarrow$

$$(n+1)\cdot n + n \leq (n+1)^2$$

$$\Rightarrow \frac{1}{n+1} + \frac{1}{(n+1)^2} \leq \frac{1}{n} \Rightarrow 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}.$$

From this, we can use the inductive hypothesis to see

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}.$$

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⑥ Base case: $P \equiv R_1$ implies $P \to Q \equiv R_1 \to Q$ by substitution.

Inductive step: Let $P \equiv R_1 \vee \ldots \vee R_{k+1}$. Denote

$P' \equiv R_1 \vee \ldots \vee R_k$, so $P \equiv P' \vee R_{k+1}$.

Then $P \to Q \equiv (P' \vee R_{k+1}) \to Q \equiv (P' \to Q) \wedge (R_{k+1} \to Q)$

by pf by cases for $k=2$, proved in lecture notes

$\equiv (R_1 \to Q) \wedge (R_2 \to Q) \wedge \ldots \wedge (R_{k+1} \to Q)$

by induction hypothesis.

(7) The proof fails because the base case is <u>false</u>:

$$\sum_{i=0}^{1} 2^i = 1+2 = 3 \neq 4 = 2^{1+1}$$

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(8) The proof fails in the inductive step when $n=1$ and $n+1=2$. In this case, the sets of horses $h_1, \dots, h_n$ and $h_2, \dots, h_{n+1}$ are just h_1 and h_2, and have no members in common.