① Proof by cases. n perfect square $\Rightarrow$ $n = k^2$, $k \in \mathbb{Z}$.

k could have remainder 0, 1, 2, or 3 when div. by 4.

$k = 4l$: then ~~k~~ $n = k^2 = 16l^2$, remainder 0

$k = 4l+1 \Rightarrow n = k^2 = 16l^2 + 8l + 1$, remainder 1.

$k = 4l+2 \Rightarrow n = k^2 = 16l^2 + 16l + 4$, remainder 0.

$k = 4l+3 \Rightarrow n = k^2 = 16l^2 + 24l + 9$, remainder 1.

---

② Contrapositive: WTS if every bin contains at most one object, then $m \le n$.

Well, the most possible objects would be if each bin had the max #, that is to say one object per bin.

Thus $m \le n$.

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③ Cases: $x < -2$, $-2 \le x \le 3$, $x > 3$.

~~If~~ ~~x < -2~~ Denote $|x+2| - |x-3|$ as $F(x)$.

If $x < -2$, $F(x) = -x - 2 + x - 3 = -5$, so $-5 \le F(x) \le 5$.

If $x > 3$, $F(x) = x + 2 - x + 3 = 5$, so ~~F(x)~~ $-5 \le F(x) \le 5$.

If $-2 \le x \le 3$, then $F(x) = x + 2 + x - 3 = 2x - 1$.

On the other hand, $-2 \le x \le 3 \rightarrow -4 \le 2x \le 6$

$\rightarrow -5 \le 2x - 1 \le 5$

$\rightarrow -5 \le F(x) \le 5$ $\blacksquare$

(4) Contrapositive: WTS that if $a < \sqrt{c}$ and $b < \sqrt{c}$ then $ab < c$. If $a < \sqrt{c}$ and $b < \sqrt{c}$, then $a \cdot b < \sqrt{c} \cdot \sqrt{c} = c$.

---

(5) This does not prove THI because we have only verified the claim for <u>one</u> positive integer. The negation of

$$\forall n \in \mathbb{N}; \sum_{i<n} i \neq n \quad \text{is} \quad \exists n \in \mathbb{N} : \sum_{i<n} i = n \ldots$$

we failed to "flip" the quantifier.

↳ The result, incidentally, is false: For $n = 3$, $\underline{1 + 2 = 3}$.

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(6) P: $x^2y = 2x+y$ $\quad Q$: $y \neq 0$ $\quad R$: $x \neq 0$

WTS $\quad P \to (Q \to R)$

<u>Direct</u>: Assume $x^2y = 2x+y$ and $y \neq 0$. Then, since $y \neq 0$ and

$y \cdot x^2 - 2 \cdot x - y = 0$

We can use the quadratic formula to conclude

$$x = \frac{2 \pm \sqrt{4 + 4y^2}}{2y} = \frac{1 \pm \sqrt{1 + y^2}}{y}.$$

Note $1 + \sqrt{1 + y^2} > 0$ and $1 - \sqrt{1+y^2} < 0$

so neither of the potential values of x can be 0.

<u>Contrapositive:</u> $\neg(Q \to R) \to \neg P$

$$\equiv \quad Q \wedge \neg R \to \neg P$$

WTS $\quad y \neq 0$ and $x = 0 \quad \Rightarrow \quad x^2y \neq 2x+y$.

Since $x=0$, $\quad x^2y = 0$.
Since $y \neq 0$ and $x = 0$, $\quad 2x+y = y \neq 0$.
Thus $\quad x^2y \neq 2x+y$.

<u>Contradiction:</u> $\quad P \wedge \neg(Q \to R) \equiv P \wedge (Q \wedge \neg R)$

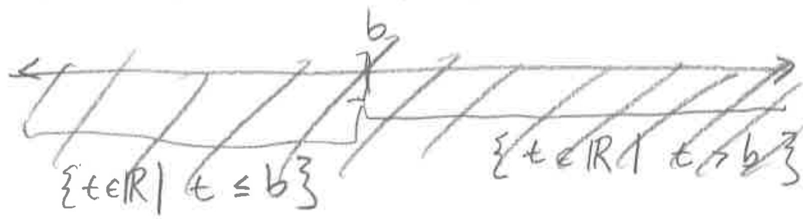
FTSOC, assume $\quad x^2y = 2x+y$ and $y \neq 0$ and $x = 0$.

Since $x=0$, $\quad \cancel{2x+y \neq y,} \quad 0 = x^2y = 2x+y = y$, so $y = 0$. But this contradicts $y \neq 0$, so we are done.

---

(7) $\quad P: \forall x > b: a \leq x. \quad Q: a \leq b$

<u>Direct:</u> $P \to Q$: ~~Consider partitioning the real line~~



Note that $a \leq b + \varepsilon$ for every $\varepsilon > 0$. Since ε can be taken arbitrarily small, we have that

$a \leq b$.

Contrapositive $\neg Q \to \neg P$.

WTS $a > b \to \exists\, x > b : a > x$.

Since $a > b$, we have $a > \frac{a+b}{2} > b$, so we have a number $x = \frac{a+b}{2}$ st $a > x$ and $x > b$.

Contradiction: $P \wedge \neg Q$.

FTSOC, assume that $\forall x > b$ we have $a \le x$, but $a > b$.

Consider $x = \frac{a+b}{2}$. since $x > b$, we must have $a \le x$.

But $a > \frac{a+b}{2}$ since $a > b$, so we have a contradiction.

---

(8) Proof by contrapositive only applies to implicative statements. We proceed by contradiction.

FTSOC, assume that $\forall i,\ y_i < A$ or $\forall j,\ y_j > A$.

case 1: $\forall i,\ y_i < A$. Well, $A = \frac{1}{n} \cdot \sum_{i=1}^{n} y_i < \frac{1}{n} \sum_{i=1}^{n} A = A$,

so $A < A$ ✱.

case 2: $\forall j,\ y_j > A$. Well, $A = \frac{1}{n} \sum_{j=1}^{n} y_j > \frac{1}{n} \sum_{j=1}^{n} A = A$,

so $A > A$ ✱.

(9) Let $p(x,y)$: $x = y$, $q(x)$: x is even

$\forall x \, \exists y \; p(x,y) \wedge q(x) \equiv$ For every x, there is y st

$x = y$ and x is even.

Negation: $\exists x \, \forall y \; \neg p(x,y) \vee \neg q(x)$

$\equiv$ There exists x st for all y, $x \neq y$ or x is not even.

---

(10) The quantifiers are reversed.

(a) $\forall x \, \exists y, \; p(x,y)$ is true. For $x \in \mathbb{N}$, we can take

$y := \frac{1}{x} \in \mathbb{Q}$ to make $xy = x \cdot \frac{1}{x} = 1$.

(b) $\exists y \, \forall x \; p(x,y)$ is false. AFSOC that it is true,

and there is $y \in \mathbb{Q}$ st $xy = 1$ $\forall x \in \mathbb{Z}$. Then

$y = 1 \cdot y = 1$ and so $2y = 2$, but also

$\cancel{2y = 2y}$ $2 \cdot y = 1$, so $1 = 2$. $\ast$.