21-127 HW2 SOL'NS

① If $a|b$, then $b = ka$ for some $k \in \mathbb{Z}$.
If $b|c$, then $c = lb$ for some $l \in \mathbb{Z}$.
Thus $c = lb = l(ka) = (lk)a$, where $lk \in \mathbb{Z}$.
Thus $a|c$.

---

② $m+n$ odd $\Leftrightarrow$ only one of n, m is even

"$\Leftarrow$" WLOG, let n be even, m be odd, ie
$n = 2k$, $m = 2l+1$. Then
$m+n = 2k + 2l + 1 = 2(k+l)+1 \Rightarrow m+n$ odd.

"$\Rightarrow$" Contra positive: 2 ~~cased~~ cases, n, m even or n, m odd

Case 1: $n = 2k$, $m = 2l \Rightarrow n+m = 2(k+l)$ is even

Case 2: $n = 2k+1$, $m = 2l+1 \Rightarrow n+m = 2(k+l)+2$ is even.

Since (n,m even or n,m odd) $\Rightarrow$ ~~(~~ $n+m$ even, by contrapositive
$n+m$ odd $\Rightarrow$ one even, one odd.

---

③ ~~Claim~~ Let x solve $\sqrt{x+1} + \sqrt{x-3} = 4$, so

$\sqrt{x+1} = 4 - \sqrt{x-3} \Rightarrow x+1 = 16 - 8\sqrt{x-3} + (x-3)$

$\Rightarrow 4 - 16 = -8\sqrt{x-3} \Rightarrow$ ~~$\frac{-14}{-8} = \sqrt{x-3} \Rightarrow$~~ $\frac{12}{8} = \sqrt{x-3} \Rightarrow \frac{3}{2} = \sqrt{x-3}$

$\Rightarrow \frac{9}{4} = x-3$

$\Rightarrow x = \frac{21}{4}$.

Thus if x is a soln, then $x = 21/4$. Lets check that it is a real soln: $\frac{21}{4}+1 = \frac{25}{4}$, $\frac{21}{4} - 3 = \frac{9}{4}$

$\Rightarrow \sqrt{\frac{21}{4}+1} + \sqrt{\frac{21}{4}-3} = \frac{5}{2} + \frac{3}{2} = 4$ ✓

④ Let p be a positive prime integer. FTSOC, say $\sqrt{p} \in \mathbb{Q}$, ie $\exists\, m, n \in \mathbb{Z}$ st $\frac{m}{n} = \sqrt{p}$ is in reduced terms.

Then $m^2 = pn^2$, so $p \mid m^2$. Since p is prime, $p \mid m$.

But if $p \mid m$, ~~then $p^2 \mid m^2$, so m^2~~ we can write $m = p \cdot k$. Thus $p^2 k^2 = pn^2 \Rightarrow n^2 = pk^2$, so $p \mid n^2$ and in fact $p \mid n$. But $p \mid m$ and $p \mid n$ contradicts the claim that $\frac{m}{n}$ is in reduced terms. We conclude $\sqrt{p} \notin \mathbb{Q}$.

---

⑤ Let $x \in \mathbb{Q}$, $y \notin \mathbb{Q}$. FTSOC, assume $x + y \in \mathbb{Q}$.

But $x \in \mathbb{Q}$, $x + y \in \mathbb{Q} \Rightarrow (x+y) - x \in \mathbb{Q}$

$\Rightarrow y \in \mathbb{Q}$ ※.

---

⑥ Proof by contrapositive: If l, k are not both even, show $ak^2 + bkl + cl^2 \neq 0$.

2 cases: Both odd, or one odd, one even.

Both odd: Since a, b, c odd, ak^2, bkl, cl^2 are all odd, so their sum is odd, which cannot be zero.

One odd, one even: WLOG, let k be odd, l be even.

Then, ak^2 is odd, blk, cl^2 even

$\Rightarrow$ their sum is odd, which cannot be zero

(7) Let n have d digits $b_0, \dots, b_{d-1}$.

Then $$n = \sum_{i=0}^{d-1} b_i \, 10^i = \sum_{i=0}^{d-1} b_i + \sum_{i=0}^{d-1} b_i \left(10^i - 1\right).$$

But $$10^i - 1 = 9 \cdot \left(\sum_{k=0}^{i-1} 10^k \right) \quad \forall i \geq 1 \quad \text{(Geometric series)}$$

so $$n = \left(\sum_{i=0}^{d-1} b_i \right) + 9 \cdot M \quad \text{for some } M \in \mathbb{N}.$$

Thus n is divisible by 9 if and only if $\sum_{i=0}^{d-1} b_i$ is divisible by 9.

---

(8) FTSOC, assume there are finitely many primes $p_1, \dots, p_n$. Let $P = \prod_{i=1}^{n} p_i$ denote their product,

so $p_i \mid P \ \forall i = 1, \dots, n$. By Ex. 11, we have $p_i \nmid (P+1) \ \forall i = 1, \dots, n$. But $P+1$ is an integer, and every integer is divisible by some prime. $*$.