Math 127 Homework

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Complete the following problems. Fully justify each response.

1. Let $a, b \in \mathbb{Z}$, $n \in \mathbb{N}$ with $n \geq 1$, and $k, \ell \in \mathbb{N}$ with $k \equiv \ell \pmod{n}$ and $a \equiv b \pmod{n}$.
   (a) Is it true that $a^k \equiv b^k \pmod{n}$? If so, prove it. If not, provide a counterexample.
   (b) Is it true that $a^k \equiv a^\ell \pmod{n}$? If so, prove it. if not, provide a counterexample.
2. Let $a \in \mathbb{Z}$, and let $n \in \mathbb{N}$ with $n \geq 1$. Suppose that $a \perp n$. Show that u, u' are both multiplicative inverses for a if and only if u is a multiplicative inverse for a and $u \equiv u' \pmod{n}$.
3. Let p be a positive prime, and $k \in \mathbb{N}$ with $k \geq 1$. Prove that $\varphi(p^k) = p^k - p^{k-1}$.
4. Read the proof of Theorem 3.3.49 and Example 3.3.51. Then prove that for any $b \in \mathbb{N}$ with $b \geq 2$, and $a \in \mathbb{N}$, a is divisible by $b - 1$ if and only if the sum of the base b digits of a is divisible by $b - 1$.
5. For each of the following functions, determine if it is injective, surjective, both, or neither. Prove that your answers are correct.
   (a) $f : \mathbb{Z} \to \mathbb{N}$, $f(x) = x^2$.
   (b) $g : \mathbb{N} \to \mathbb{Z}$, $g(x) = x^2$.
   (c) $h : \mathbb{R} \to \mathbb{Z}$, $h(x) = \lfloor x \rfloor$
   (note: $\lfloor x \rfloor$ is the number you get by rounding x down to the nearest integer. Formally, we define

   $$\lfloor x \rfloor = \max\{y \in \mathbb{Z} \mid y \leq x\}.$$

   You may be reasonably skeptical that such a number exists, since we cannot apply the Well-Ordering Principle here.... so if you are skeptical, prove it.)
   (d) $f : \mathbb{N} \to \mathbb{Z}$, $f(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ -\frac{x+1}{2} & x \text{ is odd} \end{cases}$
6. Let $f : X \to Y$ and $g : Y \to Z$ be bijective functions. Prove that $g \circ f$ is also bijective. Is the converse true?