Math 127 Homework

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Complete the following problems. Fully justify each response.

NOTE: due to the Spring Break, this homework set is a bit longer than is typical. You only need to turn in those problems marked with (*).

1. In lecture we defined a finite set X to be of size n if there exists a bijection $f : [n] \to X$.

   Prove that if X, Y are finite, and there exists a bijection $f : X \to Y$, then $|X| = |Y|$.

2. (*) Let X be a finite set, and suppose there is a surjection $f : X \to Y$. Prove that $|X| \geq |Y|$.

3. (*) Let
$$X_2 = \{n \mid 1 \leq n \leq 200, n = k^2 \ \exists k \in \mathbb{Z}\},$$
$$X_3 = \{n \mid 1 \leq n \leq 200, n = k^3 \ \exists k \in \mathbb{Z}\},$$
   and
$$X_4 = \{n \mid 1 \leq n \leq 200, n = k^4 \ \exists k \in \mathbb{Z}\}.$$
   Determine $|X_2 \cup X_3 \cup X_4|$.

4. Prove Theorem 4.2.12: De Morgan's Law for sets (finite version).

5. You decide to go to Chipotle for a burrito. Chipotle has many options: 2 choices of rice, 2 choices of beans, 5 choices of meat, 4 choices of salsa, and 5 other toppings (including queso and guacamole).

   How many different burritos could you order having exactly 1 rice, 1 bean, 1 meat, 2 salsas, and 2 other toppings?

6. (*) Let $X = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \{0,1\} \forall i\} = \{0,1\}^n$. These are sometimes called bitstrings of length n.

   (a) Show that there is a bijection between X and $\{f : [n] \to \{0,1\}\}$, the set of functions from $[n]$ to $\{0,1\}$

   (b) Show that there is a bijection between X and $\mathcal{P}([n])$.

   (c) Determine $|X|$.

7. (*) Let X and Y be finite sets. Define $X^Y = \{f : Y \to X\}$, the set of functions from Y to X. Prove that $|X^Y| = |X|^{|Y|}$.

8. (*) Let $n, k \in \mathbb{N}$ with $n \geq k$. Prove, by counting in 2 ways, that $k\binom{n}{k} = (n-k+1)\binom{n}{k-1}$.

9. Let $n, m \in \mathbb{N}$ with $m \leq n$. Prove that

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m}.$$

10. (*) How many subsets of [20] contain a multiple of 4? Prove that your answer is correct.

11. (*) Let $f : X \to Y$ be a bijection. Prove that X is countably infinite if and only if Y is countably infinite.

12. Show directly (without using Proposition 4.3.10) that $\mathbb{N} \times \mathbb{N}$ is countably infinite. (If you cannot derive a bijection, you may look to Problem 4.3.8 for one).

13. (*) Let X be a finite set. Show that $\mathbb{N}^X = \{f : X \to \mathbb{N}\}$ is countably infinite.