Math 127 Homework

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Complete the following problems. Fully justify each response.

1. Let $a, b \in \mathbb{Z}$. Prove that if d, d' are both gcds of a and b, then $d = \pm d'$.
2. Let $a, b \in \mathbb{Z}$, and let $d = \gcd(a, b)$. Prove that $\frac{a}{d}$ and $\frac{b}{d}$ are coprime.
3. Let $a, b \in \mathbb{Z}$. Prove that there exists a unique positive least common multiple of a and b. (Note: here you must prove both existence and uniqueness.)
4. Let $p \in \mathbb{Z}$. Prove that the following are equivalent[1]:
   (a) p is irreducible.
   (b) The only divisors of p are $\pm 1, \pm p$
   (c) p is prime (under the definition in Section 3.2, that p is prime whenever $p|ab \Rightarrow p|a \vee p|b$).
5. Suppose $p_1, p_2, \ldots, p_r \in \mathbb{Z}$ are primes. Let $a = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$, and let $b = p_1^{\ell_1} p_2^{\ell_2} \ldots p_r^{\ell_r}$, where $k_1, k_2, \ldots k_r, \ell_1, \ell_2, \ldots, \ell_r$ are nonnegative integers. Prove that $\gcd(a, b) = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}$, where $m_i = \min\{k_i, \ell_i\}$ for all $1 \leq i \leq r$.
6. Prove that for all $n \geq 2$, there exists a prime in the set
$$\{k \in \mathbb{Z} \mid n \leq k \leq n!\}.$$
   (Hint: consider the divisors of $n! - 1$. Can they be in the set $\{1, 2, \ldots, n\}$?)

[1] To prove that a series of propositions are equivalent, one must show that all the propositions are logically equivalent. This amounts to an if and only if type statement between each pair of propositions. However, one can avoid having to prove THAT much stuff, by instead proving something of the type $a \Rightarrow b \Rightarrow c \Rightarrow a$. This would allow us to deduce logical equivalence, as, for example, $b \Rightarrow c$ and also $c \Rightarrow a \Rightarrow b$ implies that $c \Rightarrow b$, allowing us to thus deduce a logical equivalence between b and c without having to prove it directly. If you are confused about this, please ask me or your TA.