21-127 Exam 1 Review Materials

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Major topics for your first exam include:

- Familiarity with the important number sets we use: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
- Divisibility, including the Division Theorem, remainders, and essential definitions related to division
- Base expansion for integers
- Basics of proofwriting: how to structure proofs by contradiction, contrapositive, cases, biconditional proofs
- Induction, both strong and weak: you should know both how to use these tools and why/how they work.
- Fluency in the notation and structure of propositional formulae, including building truth tables to establish logical equivalence and translating mathematical statements into propositional formulae
- De Morgan's Laws for propositional formulae: both for conjunction/disjunction and universal/existential quantification
- Basics of sets and subsets, proving subset containment and equality, power set, unions and intersections

Some practice problems:

1. Let $a, u, b, v, d \in \mathbb{N}$. For each of the following, determine if it is true or false. Prove that your answer is correct.
   (a) If $d|a$ and $d|b$, then $d|(au + bv)$.
   (b) If $d|(au + bv)$ then $d|a$ and $d|b$.
   (c) If $d|a$ and d does not divide b or v, then d does not divide $au + bv$.
   (d) If d does not divide any of a, u, b, v, then d does not divide $au + bv$.
2. Prove that $\sqrt{3}$ is irrational.
3. Suppose $p(x)$ is a polynomial that can be written as
$$p(x) = (x - a_1)(x - a_2) \dots (x - a_n).$$
   Prove that α is a root of $p(x)$ if and only if $\alpha = a_i$ for some i with $1 \leq i \leq n$.
4. Find all real solutions to the equation
$$\sqrt{x + 10} + \sqrt{x + 5} = 5.$$
   Prove that your answer is correct.

5. Let $X = \{x \in \mathbb{R} \mid x > 0\}$. Prove that X does not contain a minimum element.

6. Let $a, b \in \mathbb{R}$ be real numbers, satisfying $a^2 - 4b \neq 0$. Let α, β be the distinct roots of the polynomial $p(x) = x^2 + ax + b$. Prove that there exists a real number c such that $\alpha - \beta = c$ or $\alpha - \beta = ci$.

7. Let p be prime, and $a, b \in \mathbb{Z}$. Prove that $p|ab$ if and only if $p|a$ or $p|b$.

8. Prove that for any $n \in \mathbb{N}$, $\displaystyle\sum_{k=0}^{n} k^3 = \left(\sum_{k=0}^{n} k\right)^2$.

9. Suppose $a_{n+1} = 5a_n - 6a_{n-1}$ for any $n \geq 1$, with $a_0 = 1$ and $a_1 = 1$. Prove that $a_n = 2^{n+1} - 3^n$ for all $n \in \mathbb{N}$.

10. Let $n \in \mathbb{N}$, with $n \geq 1$. Prove that $11^n - 6$ is divisible by 5.

11. Let $n \in \mathbb{N}$. Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

12. Prove that for all $n \geq 4$, $n! > 2^n$.

13. Let $n, m \in \mathbb{N}$, with $n, m \geq 1$. A chocolate bar is made up of an $n \times m$ grid of squares. You break the chocolate bar into pieces by iteratively breaking along a grid line. How many times must you make a break? Prove that your answer is correct.

14. Prove that $(\neg(p \wedge q)) \wedge r$ is logically equivalent to $\neg((p \vee (\neg r)) \wedge (q \vee (\neg r)))$.

15. Prove that $(p \implies q) \wedge (\neg p \implies q)$ is logically equivalent to q.

16. Suppose that x and y both have the range of real numbers. Explain the difference between the following two statements.
    - $\forall x, \exists y, x^2 = y$
    - $\exists y, \forall x, x^2 = y$

17. Write the following statement as a propositional formula. Then prove it.

    Let $n \in \mathbb{N}$. If n^2 is divisible by 4 and n^3 is divisible by 27, then n is divisible by 2 and n is divisible by 3.

18. Write the following statement as a propositional formula. Then prove it.

    There is no integer value of x satisfying $0x = 1$.

19. (a) Let $\mathcal{U}$ be the universe of even numbers. Write the implied list $\{4, 16, 36, 64, \dots\}$ in the form $\{x \mid p(x)\}$ for some propositional variable $p(x)$.

    (b) How is the set $\{x \in \mathbb{Z} \mid x = k^2 \text{ for some } k \in \mathbb{Z}\}$ different from the set in part (a)?

    (c) Let $\mathcal{U}$ be the universe of even numbers. Let

    $$A = \{x \in \mathcal{U} \mid x = k^2 \text{ for some } k \in \mathbb{Z}\}$$

    and let

    $$B = \{x \in \mathbb{Z} \mid x \text{ is divisible by 4 and } x/4 = k^2 \text{ for some } k \in \mathbb{Z}\}.$$

    Prove that $A = B$.

20. Let $X = \{1, 3, 4, 5, 7, 8\}$ and let $Y = \{2, 4, 5\}$. Write, as lists, the sets $X \cap Y$, $X \cup Y$, and $\mathcal{P}(Y)$.

21. Let $n \in \mathbb{N}$. Prove, by induction, that if A is a set containing exactly n elements, then $\mathcal{P}(A)$ contains exactly 2^n elements.