Math 101 Homework

Mary Radcliffe

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Complete the following problems. Fully justify each response.

1. Let $X = \{1, 2, 3, \ldots, 2n\}$. Let $\mathcal{F}$ be the collection of subsets of X that have even size. For example, if $X = \{1, 2, 3, 4\}$, then

$$\mathcal{F} = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3,4\}\}.$$

   Is $\mathcal{F}$ a σ-field? If yes, prove it. If not, which properties does it satisfy?

2. Let X be a set, and define

$$\mathcal{F} = \{S \subset X \mid \text{either } S \text{ is at most countable or } S^c \text{ is at most countable}\}.$$

   Prove that $\mathcal{F}$ is a σ-field. You may use the fact that the union of countably many countable sets is countable.

3. In the definition of a σ-field, we gave some properties in terms of set operations. Here are some more set operations we can do in a σ-field.

   Suppose $\mathcal{F}$ is a σ-field. Prove each of the following:

   (a) $\emptyset \in \mathcal{F}$

   (b) If $A_1, A_2, A_4, \cdots \in \mathcal{F}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$. (Hint: Use De Morgan's Law)

   (c) If $A, B \in \mathcal{F}$, then $A \backslash B \in \mathcal{F}$.

4. Suppose that μ is a measure on a σ-field $\mathcal{F}$.

   (a) Prove that if A, B are disjoint sets in $\mathcal{F}$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

   (b) Prove that if $A_1, A_2, \cdots \in \mathcal{F}$ are pairwise disjoint, then

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$