On the Irregular Chromatic Number of a Graph

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Abstract

For a graph G and a proper coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of the vertices of G for some positive integer k, the color code of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple code$(v) = (a_0, a_1, \ldots, a_k)$, where a_0 is the color assigned to v and, for $1 \leq i \leq k$, a_i is the number of the vertices of G adjacent to v that are colored i. The coloring c is irregular if distinct vertices have distinct color codes and the irregular chromatic number $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k-coloring. We study irregular colorings of cycles and trees. The irregular chromatic numbers of the cycle and path of order n are determined for $3 \leq n \leq 100$. For each integer $n \geq 2$, let $D_T(n)$ and $d_T(n)$ be the maximum and the minimum irregular chromatic numbers among all trees of order n, respectively. It is shown that $D_T(n) = n$ and the values of $d_T(n)$ are determined for $3 \leq n \leq 100$. We investigate how the irregular chromatic number of a graph can be affected by removing a vertex or an edge from the graph. Also, we survey the results, conjectures, and problems on this topic.

Key Words: irregular coloring, irregular chromatic number.

AMS Subject Classification: 05C15.

1 Introduction

A (proper) *coloring* of a graph G is a function $c : V(G) \to \mathbf{N}$ having the property that $c(u) \neq c(v)$ for every pair u, v of adjacent vertices of G, where $\mathbf{N}$ is the set of positive integers. A k*-coloring* of G uses k colors. The *chromatic number* $\chi(G)$ of G is the minimum positive integer k for which there is a k-coloring of G. For a positive integer k and a proper coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of the vertices of a graph G, the *color code* of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple

$$\text{code}_c(v) = (a_0, a_1, \ldots, a_k),$$

where a_0 is the color assigned to v (that is, $a_0 = c(v)$)) and, for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored i. If the coloring c is clear, we write $\text{code}_c(v)$ as $\text{code}(v)$ or simply, $\text{code}(v) = a_0 a_1 a_2 \cdots a_k$. Therefore, if $a_0 = j$ for some j with $1 \leq j \leq k$, then $a_j = 0$ and $\sum_{i=1}^{k} a_i = \deg_G v$. The coloring c is called *irregular* if distinct vertices have distinct color codes and the *irregular chromatic number* $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k-coloring. An irregular k-coloring with $\chi_{ir}(G) = k$ is a *minimum irregular coloring.* Since every irregular coloring of a graph G is a coloring of G, it follows that $\chi(G) \leq \chi_{ir}(G)$. The following useful observation was stated in [12]. The *neighborhood* of a vertex u in a graph G is $N(u) = \{v \in V(G) : uv \in E(G)\}$.

Observation 1.1 *Let c be a (proper) coloring of the vertices of a nontrivial graph G and let u and v be two distinct vertices of G.*

(a) *If $c(u) \neq c(v)$, then $\text{code}(u) \neq \text{code}(v)$.*

(b) *If $\deg_G u \neq \deg_G v$, then $\text{code}(u) \neq \text{code}(v)$.*

(c) *If c is irregular and $N(u) = N(v)$, then $c(u) \neq c(v)$.*

Irregular colorings were introduced in [12] and studied further in [13], inspired by the problem in graph theory that concerns finding means to distinguish all the vertices of a connected graph. This problem has received increased attention during the past 35 years (see [1, 2, 3, 4, 5, 6, 8, 9, 10, 15] for example). We refer to the book [7] for graph theory notation and terminology not described in this paper.

2 Some Known Results

In this section, we present some known results on irregular chromatic numbers of graphs. The following theorems 2.1-2.3 were established in [12, 13] and will be useful to us.

Theorem 2.1 *For every pair a, b of integers with $2 \leq a \leq b$, there is a connected graph G with $\chi(G) = a$ and $\chi_{ir}(G) = b$.*

Theorem 2.2 *If a nontrivial connected graph G has an irregular k-coloring, then G contains at most $k\binom{r+k-2}{r}$ vertices of degree r.*

Since every nontrivial graph G has at least two vertices of the same degree, any irregular coloring of G must use at least two distinct colors. Furthermore, the coloring of a graph G that assigns distinct colors to distinct vertices of G is irregular and so $\chi_{ir}(G)$ always exists. Therefore, if G is a nontrivial graph of order n, then $2 \leq \chi_{ir}(G) \leq n$. Connected graphs of order $n \geq 2$ having irregular chromatic number 2 or n are characterized in [12].

Theorem 2.3 *Let G be a nontrivial connected graph of order n. Then*

(a) *$\chi_{ir}(G) = 2$ if and only if n is even and $G \cong F_n$, where $n = 2k$ for some positive integer k and F_n is the bipartite graph with partite sets $X = \{x_1, x_2, \cdots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ such that $\deg x_i = \deg y_i = i$ for $1 \leq i \leq k$;*

(b) *$\chi_{ir}(G) = n$ if and only if G is a complete multipartite graph.*

Furthermore, for each pair k, n of integers with $2 \leq k \leq n$, there exists a connected graph of order n having irregular chromatic number k if and only if $(k, n) \neq (2, n)$ for any odd integer n.

A well-known result involving a graph and its complement provides upper and lower bounds for both the sum and the product of the chromatic numbers of a graph and its complement. In particular, for every graph G of order n,

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1 \text{ and } n \leq \chi(G)\chi(\overline{G}) \leq \left(\tfrac{n+1}{2}\right)^2,$$

which are called the *Nordhaus-Gaddum inequalities* and are due to Nordhaus and Gaddum [11]. Nordhaus-Gaddum inequalities for the irregular chromatic number of a graph and its complement were established in [13], which we state as follows.

Theorem 2.4 *If G is a graph of order n, then*

$$2\sqrt{n} \leq \chi_{ir}(G) + \chi_{ir}(\overline{G}) \leq 2n \text{ and } n \leq \chi_{ir}(G)\chi_{ir}(\overline{G}) \leq n^2.$$

Furthermore, each of these four bounds is sharp.

It is well-known that if H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$. However, this is not true for the irregular chromatic number of a graph. For example, the wheel $W_5 = C_5 + K_1$ has irregular chromatic number 4, where an irregular 4-coloring is shown in Figure 1; while the subgraph (not induced) $K_{1,5}$ of W_5 has irregular chromatic number 6 by Theorem 2.3. A minimum irregular coloring of $K_{1,5}$ is also shown in Figure 1. In fact, more can be said.

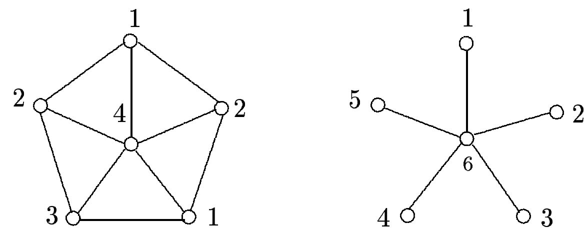

Figure 1: Minimum irregular colorings of W_5 and $K_{1,5}$

Proposition 2.5 *For every pair a, b of integers with $2 \le a \le b$ there exists a graph $G_{a,b}$ containing an induced subgraph $H_{a,b}$ such that*

$$\chi_{ir}(G_{a,b}) = a \text{ and } \chi_{ir}(H_{a,b}) = b.$$

Proof. For $a = 2$, let $G_{2,b} = F_{2b}$ be the bipartite graph with partite sets $X = \{x_1, x_2, \cdots, x_b\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$ such that $\deg x_i = \deg y_i = i$ for $1 \le i \le b$ (as described in Theorem 2.3). Then $\chi_{ir}(G_{2,b}) = 2$. Observe then that x_b is adjacent to all vertices in Y and x_{b-1} is adjacent to all vertices in $Y - \{y_1\}$. In general, for each integer j with $1 \le j \le b$,

$$N(x_j) = \{y_i : \quad b - j + 1 \le i \le b\}.$$

For $a \ge 3$, let $G_{a,b}$ be the graph obtained from F_{2b} and K_a by identifying x_b of F_{2b} with a vertex of K_a. Since $G_{a,b}$ contains K_a as a subgraph, $\chi_{ir}(G_{a,b}) \ge a$. On the other hand, we define a coloring c of $G_{a,b}$ by assigning color 1 to each vertex in X, color 2 to each vertex in Y, and the $a - 1$ distinct colors $2, 3, \ldots, a$ to the remaining $a - 1$ vertices of K_a. Since c is an irregular a-coloring of $G_{a,b}$, it follows that $\chi_{ir}(G_{a,b}) \le a$ and so $\chi_{ir}(G_{a,b}) = a$. Figure 2 shows the graph $G_{3,4}$ together with an irregular 3-coloring of $G_{3,4}$.

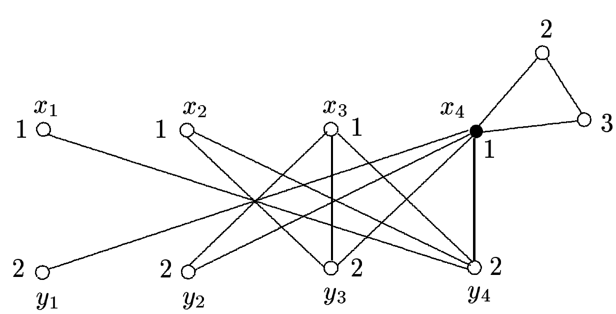

Figure 2: The graph $G_{3,4}$

For each integer $a \geq 2$, the induced subgraph

$$H_{a,b} = \langle \{x_{b-1}\} \cup N(x_{b-1}) \rangle$$

in $G_{a,b}$ is isomorphic to the star $K_{1,b-1}$ of order b and so $\chi_{ir}(H_{2,b}) = b$ by Theorem 2.3. ■

3 Irregular Colorings of Cycles

The irregular chromatic number of a cycle C_n of order $n \geq 3$ was studied in [12]. For $3 \leq n \leq 9$, $\chi_{ir}(C_n) = 4$ if n is even and $\chi_{ir}(C_n) = 3$ if n is odd. If $n \geq 10$, then $\chi_{ir}(C_n) \geq 4$ by Theorem 2.2. The following conjecture appears in [12].

Conjecture 3.1 *Let $k \geq 4$. If $(k-1)\binom{k-1}{2} + 1 \leq n \leq k\binom{k}{2}$, then*

$$\chi_{ir}(C_n) = \begin{cases} k & \text{if } n \neq k\binom{k}{2} - 1 \\ k+1 & \text{if } n = k\binom{k}{2} - 1. \end{cases}$$

It was observed that if k and n are integers satisfying the conditions in Conjecture 3.1, then $\chi_{ir}(C_n) \geq k$. Conjecture 3.1 was verified for $10 \leq n \leq 50$. The largest possible value of n for which $\chi_{ir}(C_n) = 5$ is 50 and an irregular 5-coloring of C_{50} was found in [12], which is shown in Figure 3.

In this section, we show that Conjecture 3.1 is true for $51 \leq n \leq 100$. In order to do this, we first present a useful result (see [12]) and an additional definition.

Theorem 3.2 *Let $k \geq 3$ and $n = \frac{k^2(k-1)}{2}$.*

If $\chi_{ir}(C_n) = k$, then $\chi_{ir}(C_{n-1}) = k+1$.

For an irregular coloring c of the cycle $C_n : v_1, v_2, \ldots, v_n, v_1$ of order n, define the *color sequence s_n of C_n with respect to c* as the sequence

$$s_n : c(v_1), c(v_2), \cdots, c(v_n).$$

For example, for the irregular 5-coloring of C_{50} of Figure 3, the color sequence is

$$\begin{aligned} s_{50}: \quad & 1,2,1,2,3,2,3,1,3,1,2,4,2,4,1,4,1,2,5,2,5,4,5,4, \\ & 1,5,1,5,4,3,4,3,5,4,2,3,5,3,5,1,3,5,2,3,4,1,3,4,2,5, \end{aligned} \tag{1}$$

where the vertex v_1 is indicated in Figure 3. Therefore, an irregular coloring of a cycle can be represented by its color sequence. We now show that Conjecture 3.1 holds for $51 \leq n \leq 100$.

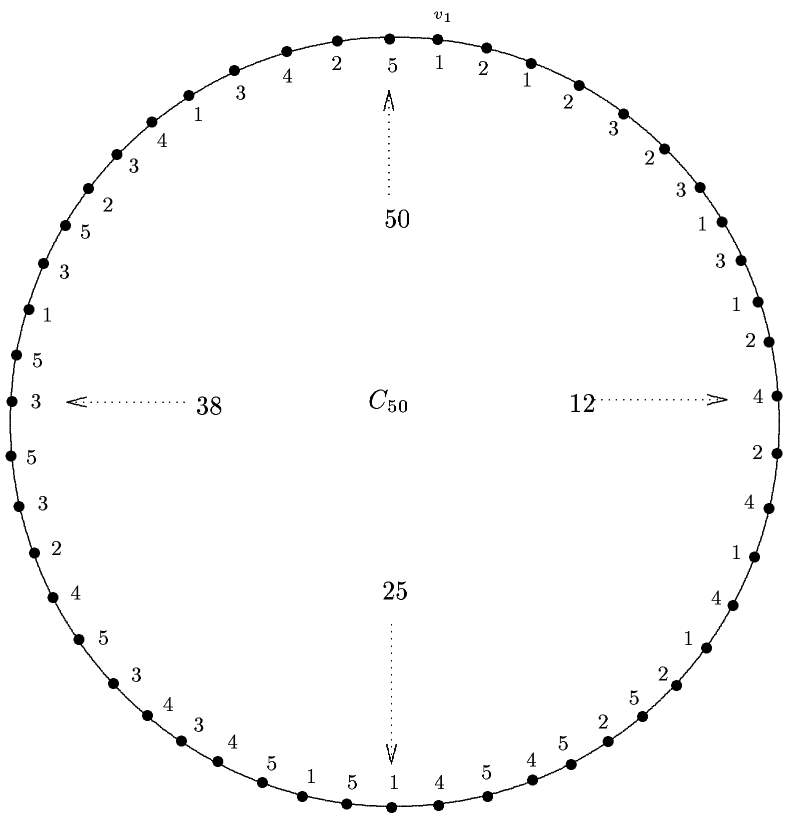

Figure 3: An irregular 5-coloring of C_{50}

Proposition 3.3 *For* $51 \le n \le 100$,

$$\chi_{ir}(C_n) = \begin{cases} 6 & \text{if } 51 \le n \le 90 \text{ and } n \ne 89 \\ 7 & \text{if } n = 89 \text{ or if } 91 \le n \le 100. \end{cases}$$

Proof. First, assume that $51 \le n \le 62$. Since the largest possible value of n for which $\chi_{ir}(C_n) = 5$ is 50, it follows that $\chi_{ir}(C_n) \ge 6$ for $n > 50$. On the other hand, with the aid of the 5-coloring of C_{50} in (1), we are able to obtain an irregular 6-coloring c_n for the cycle C_n for each integer n with $51 \le n \le 62$. Let s_{48} be the color sequence of C_{48} obtained from the color sequence s_{50} in (1) by deleting the first two terms 1, 2 in s_{50}. The color sequence s_n of C_n with respect to the irregular 6-coloring c_n for each integer n with $51 \le n \le 62$ is shown as follows:

$s_{51}:\ s_{50}, 6,$ $\qquad$ $s_{52}:\ s_{48}, 6, 1, 3, 6$

$s_{53}:\ s_{50}, 6, 1, 6,$ $\qquad$ $s_{54}:\ s_{50}, 6, 1, 3, 6,$

$s_{55}:\ s_{50}, 6, 1, 6, 2, 6,$ $\qquad$ $s_{56}:\ s_{50}, 6, 1, 3, 6, 2, 6,$

$s_{57}:\ s_{50}, 6, 1, 6, 2, 6, 3, 6,$ $\qquad$ $s_{58}:\ s_{50}, 6, 1, 3, 6, 2, 6, 3, 6,$

$s_{59}:\ s_{50}, 6, 1, 6, 2, 6, 3, 6, 4, 6,$ $\qquad$ $s_{60}:\ s_{50}, 6, 1, 3, 6, 2, 6, 3, 6, 4, 6,$

$s_{61}:\ s_{50}, 6, 1, 6, 2, 6, 3, 6, 4, 6, 5, 6,$ $\qquad$ $s_{62}:\ s_{50}, 6, 1, 3, 6, 2, 6, 3, 6, 4, 6, 5, 6.$

Therefore, $\chi_{ir}(C_n) = 6$ for $51 \leq n \leq 62$.

Next, assume that $63 \leq n \leq 90$. The largest possible value of n for which $\chi_{ir}(C_n) = 6$ is 90. In fact, $\chi_{ir}(C_{90}) = 6$ since there is an irregular 6-coloring of C_{90}, as shown in Figure 4. This coloring is constructed with the aid of a deBruijn digraph as described in [12].

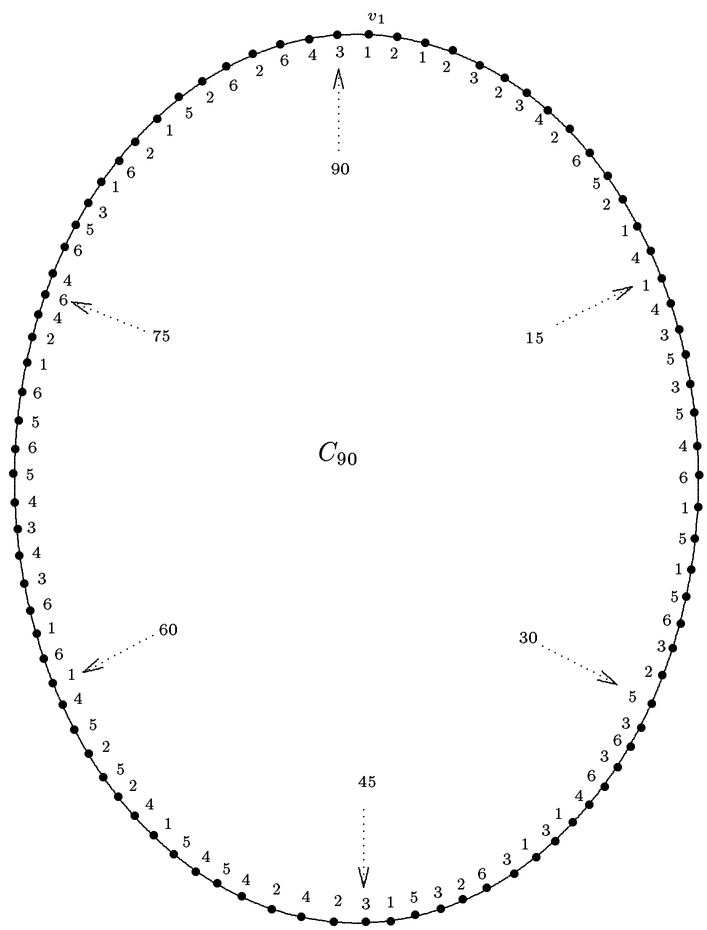

Figure 4: An irregular 6-coloring of C_{90}

Since $\chi_{ir}(C_{90}) = 6$, it then follows by Theorem 3.2 that $\chi_{ir}(C_{89}) = 7$. For the irregular 6-coloring of C_{90} of Figure 4, the color sequence is

$$\begin{array}{ll} s_{90}: & 1,2,1,2,3,2,3,4,2,6,5,2,1,4,1,4,3,5,3,5,4,6, \\ & 1,5,1,5,6,3,2,5,3,6,3,6,4,1,3,1,3,6,2,3,5,1,3, \\ & 2,4,2,4,5,4,5,1,4,2,5,2,5,4,1,6,1,6,3,4,3,4, \\ & 5,6,5,6,1,2,4,6,4,6,5,3,1,6,2,1,5,2,6,2,6,4,3, \end{array} \qquad (2)$$

where the vertex v_1 is indicated in Figure 4. In the color sequence of C_{90} in (2) there are 13 subsequences of the form i, j, i, j (or more simply $ijij$), where $1 \leq i \neq j \leq 6$, namely

$$1212, 1414, 3535, 1515, 3636, 1313, 2424, 2525, 1616, 3434, 5656, 4646, 2626. \qquad (3)$$

Successively replacing these subsequences $ijij$ in s_{90} by ij in the order described in (3), we obtain color sequences for C_n, where $64 \leq n \leq 88$ and n is even. Therefore, $\chi_{ir}(C_n) = 6$ when $64 \leq n \leq 88$ and n is even.

Figure 5 shows an irregular 6-coloring of C_{87} and so $\chi_{ir}(C_{87}) = 6$. For this irregular 6-coloring of C_{87}, the color sequence is

$$\begin{array}{ll} s_{87}: & 1,2,4,1,3,1,3,5,4,5,4,6,1,6,1,3,4,3,4,1,4,1,5,1,5,4,3,5,3,5, \\ & 6,5,6,3,6,3,4,6,4,6,2,6,2,1,2,1,4,5,6,1,5,2,5,2,6,5,2,3,2,3, \\ & 6,4,2,4,2,5,3,2,4,3,2,6,3,1,5,3,6,1,4,6,5,1,2,5,4,2,6, \end{array} \qquad (4)$$

where the vertex v_1 is indicated in Figure 5. In the color sequence of C_{87} in (4) there are 13 subsequences of the form $ijij$, where $1 \leq i \neq j \leq 6$, namely

$$1313, 5454, 6161, 3434, 1515, 3535, 6363, 4646, 6262, 2121, 5252, 2323, 4242 \qquad (5)$$

Successively replacing these subsequences $ijij$ in s_{87} by ij in the order described in (5), we obtain color sequences for C_n, where $63 \leq n \leq 85$ and n is odd. Therefore, $\chi_{ir}(C_n) = 6$ when $63 \leq n \leq 85$ and n is odd.

Finally, assume that $91 \leq n \leq 100$. Since the largest possible value of n for which $\chi_{ir}(C_n) = 6$ is 90, it follows that $\chi_{ir}(C_n) \geq 7$ for $n > 90$. In fact, $\chi_{ir}(C_n) = 7$ for $91 \leq n \leq 100$. With the aid of the 6-coloring of C_{90} in (2), we are able to obtain an irregular 7-coloring c_n of C_n for each integer n with $91 \leq n \leq 100$. Let s_{88} be the color sequence of C_{88} obtained from the color sequence s_{90} in (2) by deleting the first two terms 1, 2 in s_{90}. The color sequence s_n of C_n with respect to the irregular 7-coloring c_n for each integer n with $91 \leq n \leq 100$ is shown as follows:

$$\begin{array}{llll} s_{91}: & s_{90},7, & s_{92}: & s_{88},7,1,6,7, \\ s_{93}: & s_{90},7,1,7, & s_{94}: & s_{90},7,1,6,7, \\ s_{95}: & s_{90},7,1,7,2,7, & s_{96}: & s_{90},7,1,6,7,2,7, \\ s_{97}: & s_{90},7,1,7,2,7,3,7, & s_{98}: & s_{90},7,1,6,7,2,7,3,7, \\ s_{99}: & s_{90},7,1,7,2,7,3,7,4,7, & s_{100}: & s_{90},7,1,6,7,2,7,3,7,4,7. \end{array}$$

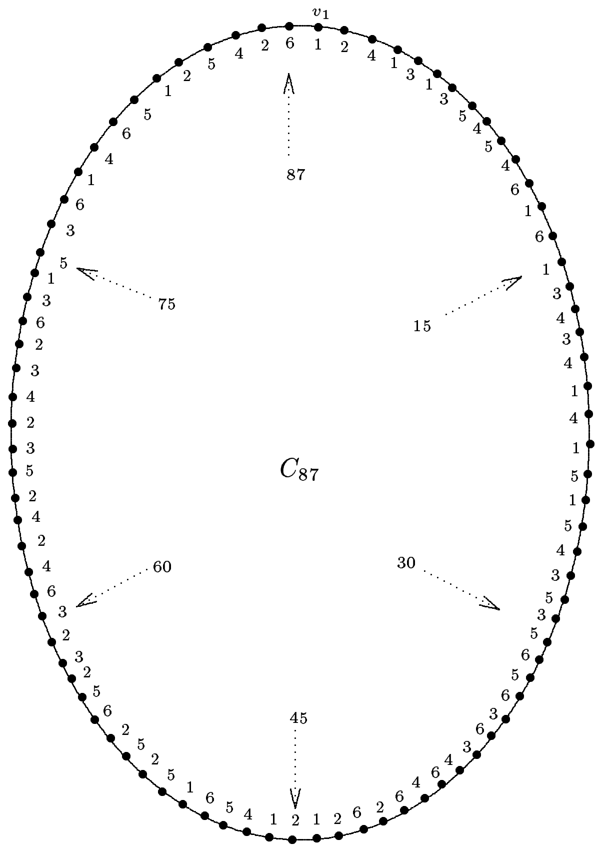

Figure 5: An irregular 6-coloring of C_{87}

Therefore, $\chi_{ir}(C_n) = 7$ for $91 \leq n \leq 100$. ■

In summary, the irregular chromatic numbers of cycles C_n of order n, where $3 \leq n \leq 100$, are as follows:

$$\chi_{ir}(C_n) = \begin{cases} 3 & \text{if } 3 \leq n \leq 9 \text{ and } n \text{ is odd} \\ 4 & \text{if } 3 \leq n \leq 9 \text{ and } n \text{ is even or} \\ & \text{if } 10 \leq n \leq 24 \text{ and } n \neq 23 \\ 5 & \text{if } n = 23 \text{ or if } 25 \leq n \leq 50 \text{ and } n \neq 49 \\ 6 & \text{if } n = 49 \text{ or if } 51 \leq n \leq 90 \text{ and } n \neq 89 \\ 7 & \text{if } n = 89 \text{ or if } 91 \leq n \leq 100. \end{cases}$$

4 Irregular Colorings of Trees

In this section we study the irregular chromatic number of a tree. By Theorem 2.3, the star $K_{1,n-1}$ of order $n \geq 3$ has irregular chromatic number n. A *double star* is a tree of diameter 3. If the two non-end-vertices of a double star have degrees a and b, respectively, then we denote this double star by $S_{a,b}$. Next we determine the irregular chromatic number of a double star.

Proposition 4.1 *For integers $a, b \geq 2$, the irregular chromatic number of the double star $S_{a,b}$ is* $\max\{a, b\}$.

Proof. Let $u, v \in V(S_{a,b})$ such that $\deg u = a$ and $\deg v = b$, where $N(u) = \{v, u_1, u_2, \ldots, u_{a-1}\}$ and $N(v) = \{u, v_1, v_2, \ldots, v_{b-1}\}$. We may assume, without loss of generality, that $a \leq b$ and so $b = \max\{a, b\}$. Since each of the $b-1$ end-vertices in $N(v)$ is adjacent to v, any irregular coloring of $S_{a,b}$ must use at least b distinct colors for the b vertices in $N(v) \cup \{v\}$ and so $\chi_{ir}(S_{a,b}) \geq b$. On the other hand, an irregular b-coloring c of $S_{a,b}$ can be defined by $c(u) = 1$, $c(v) = 2$, $c(u_i) = i + 1$ for $1 \leq i \leq a - 1$, $c(v_1) = 1$, $c(v_j) = j + 1$ for $2 \leq j \leq b - 1$. Thus $\chi_{ir}(S_{a,b}) \leq b$ and so $\chi_{ir}(S_{a,b}) = b = \max\{a, b\}$. ■

Next, we consider a special class of trees, namely paths P_n. It is easy to see that $\chi_{ir}(P_2) = 2$, $\chi_{ir}(P_3) = 3$, and $\chi_{ir}(P_4) = 2$. Thus we may assume that $n \geq 5$. The following result provides a lower bound for the irregular chromatic number of a path.

Proposition 4.2 *Let $n \geq 5$. If k is an integer such that $n > k\binom{k}{2} + 2$, then $\chi_{ir}(P_n) \geq k + 1$.*

Proof. By Theorem 2.2, if c is an irregular k-coloring of the path P_n of order n, then P_n contains at most $k\binom{k}{2}$ vertices of degree 2 and so $n \leq k\binom{k}{2} + 2$. Thus, if $n > k\binom{k}{2} + 2$, then $\chi_{ir}(P_n) \geq k + 1$. ■

For $n \geq 5$, the irregular chromatic number of a path P_n is bounded above by the irregular chromatic number of a cycle C_{n-2}, as we show next.

Proposition 4.3 *For each integer $n \geq 5$, $\chi_{ir}(P_n) \leq \chi_{ir}(C_{n-2})$.*

Proof. Let $C_{n-2} : v_1, v_2, \ldots, v_{n-2}, v_1$ be a cycle of order $n - 2$. Then P_n: $v_0, v_1, v_2, \ldots, v_{n-2}, v_{n-1}$ is a path of order n. Let c be a minimum irregular coloring of C_{n-2}. We define a coloring c^* of P_n from c by

$$c^*(v_i) = \begin{cases} c(v_{n-2}) & \text{if } i = 0 \\ c(v_1) & \text{if } i = n-1 \\ c(v_i) & \text{if } 1 \leq i \leq n-2. \end{cases}$$

Observe that the color codes of the vertices P_n are those of C_{n-2} except v_0 and v_{n-1}. Since $c^*(v_0) \neq c^*(v_{n-1})$, it follows by Observation 1.1(a) that $\text{code}_{c^*}(v_0) \neq \text{code}_{c^*}(v_{n-1})$. Furthermore, $\text{code}_{c^*}(v_0) \neq \text{code}_{c^*}(v_i)$ and $\text{code}_{c^*}(v_{n-1}) \neq \text{code}_{c^*}(v_i)$ for $1 \leq i \leq n-2$ by Observation 1.1(b). Since c is an irregular coloring of C_{n-2}, it follows that $\text{code}_{c^*}(v_i) \neq \text{code}_{c^*}(v_j)$ for $1 \leq i \neq j \leq n-2$. Thus c^* is an irregular coloring using $\chi_{ir}(C_{n-2})$ colors and so $\chi_{ir}(P_n) \leq \chi_{ir}(C_{n-2})$. ∎

In general, however, the irregular chromatic number of a path P_n of order $n \geq 5$ is unknown. By Proposition 4.2, for $n \geq 5$, if k is the unique integer such that $(k-1)\binom{k-1}{2} + 3 \leq n \leq k\binom{k}{2} + 2$, then $\chi_{ir}(P_n) \geq k$. In fact, we have the following conjecture.

Conjecture 4.4 *Let $n \geq 5$. If k is the unique integer such that*

$$(k-1)\binom{k-1}{2} + 3 \leq n \leq k\binom{k}{2} + 2,$$

then $\chi_{ir}(P_n) = k$.

Conjecture 4.4 is true for $5 \leq n \leq 100$, as we show next. By Proposition 4.3, $\chi_{ir}(P_{11}) \leq \chi_{ir}(C_9) = 3$, $\chi_{ir}(P_{26}) \leq \chi_{ir}(C_{24}) = 4$, $\chi_{ir}(P_{52}) \leq \chi_{ir}(C_{50}) = 5$, $\chi_{ir}(P_{92}) \leq \chi_{ir}(C_{90}) = 6$, and $\chi_{ir}(P_n) \leq \chi_{ir}(C_{n-2}) = 7$ for $93 \leq n \leq 100$. On the other hand, by Proposition 4.2, if $n \geq 5$, then $\chi_{ir}(P_n) \geq 3$; if $n \geq 12$, then $\chi_{ir}(P_n) \geq 4$; if $n \geq 27$, then $\chi_{ir}(P_n) \geq 5$; if $n \geq 53$, then $\chi_{ir}(P_n) \geq 6$; and if $n \geq 93$, then $\chi_{ir}(P_n) \geq 7$. Thus $\chi_{ir}(P_{11}) = 3$, $\chi_{ir}(P_{26}) = 4$, $\chi_{ir}(P_{52}) = 5$, $\chi_{ir}(P_{92}) = 6$, and $\chi_{ir}(P_n) = 7$ for $93 \leq n \leq 100$. With the aid of these observations and the proof of Proposition 4.3, we are able to determine the irregular chromatic number of P_n for $2 \leq n \leq 100$ as follows:

$$\chi_{ir}(P_n) = \begin{cases} 2 & \text{if } n = 2, 4 \\ 3 & \text{if } n = 3 \text{ or } 5 \leq n \leq 11 \\ 4 & \text{if } 12 \leq n \leq 26 \\ 5 & \text{if } 27 \leq n \leq 52 \\ 6 & \text{if } 53 \leq n \leq 92 \\ 7 & \text{if } 93 \leq n \leq 100. \end{cases}$$

Therefore, Conjecture 4.4 holds for $5 \leq n \leq 100$. In fact, with the aid of Proposition 4.3, it can be shown that if Conjecture 3.1 is true, then Conjecture 4.4 is true.

Next we investigate how large and how small the irregular chromatic number of a tree of a fixed order can be. For each integer $n \geq 2$, let $D_T(n)$ be the maximum irregular chromatic number among all trees of order n and let $d_T(n)$ be the minimum irregular chromatic number among all trees of order n. That is, if $\mathcal{T}_n$ is the set of all trees of order n, then

$$\begin{aligned} D_T(n) &= \max \{\chi_{ir}(T) : T \in \mathcal{T}_n\} \\ d_T(n) &= \min \{\chi_{ir}(T) : T \in \mathcal{T}_n\}. \end{aligned}$$

Therefore,

$$2 \leq d_T(n) \leq D_T(n) \leq n$$

for $n \geq 2$. Since $\chi_{ir}(K_{1,n-1}) = n$ for $n \geq 2$, we have the following.

Proposition 4.5 *For each integer $n \geq 2$, $D_T(n) = n$.*

Next, we study the minimum irregular chromatic number among all trees of order n. Obviously, $d_T(n) = n$ for $n = 2, 3$ and $d_T(4) = 2$ as $\chi_{ir}(P_4) = 2$. Thus, we assume that $n \geq 5$. It is known that if T is a tree of order n having n_i vertices of degree i for $1 \leq i \leq \Delta$, where Δ is the maximum degree of T, then

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \ldots + (\Delta - 2)n_\Delta. \tag{6}$$

We now establish a lower bound for $d_T(n)$, where $n \geq 5$.

Proposition 4.6 *Let $n \geq 5$ be an integer. If k is the unique integer such that*

$$\frac{k^3 - 7k + 4}{2} \leq n \leq \frac{k^3 + 3k^2 - 4k - 4}{2},$$

then $d_T(n) \geq k$.

Proof. Assume, to the contrary, that $d_T(n) \leq k - 1$. Then there exists a tree T of order n with $\frac{k^3-7k+4}{2} \leq n \leq \frac{k^3+3k^2-4k-4}{2}$ such that $\chi_{ir}(T) \leq k-1$. By Theorem 2.2, T contains at most $(k-1)(k-2)$ end-vertices and at most $(k-1)\binom{k-1}{2}$ vertices of degree 2. Suppose that T has n_i vertices of degree i for $1 \leq i \leq \Delta$, where Δ is the maximum degree of T. It then follows by (6) that

$$\begin{aligned} n &= n_1 + n_2 + \ldots + n_\Delta \\ &\leq n_1 + n_2 + n_3 + 2n_4 + 3n_5 + \ldots + (\Delta - 2)n_\Delta \\ &= n_1 + n_2 + (n_1 - 2) \\ &\leq (k-1)(k-2) + (k-1)\binom{k-1}{2} + [(k-1)(k-2) - 2] \\ &= \frac{k^3 - 7k + 2}{2} < \frac{k^3 - 7k + 4}{2}, \end{aligned}$$

which is a contradiction. ■

In general, however, the number $d_T(n)$ is not known for $n \geq 5$. In fact, we have the following conjecture.

Conjecture 4.7 *Let $n \geq 5$ be an integer. If k is the unique integer such that*

$$\frac{k^3 - 7k + 4}{2} \leq n \leq \frac{k^3 + 3k^2 - 4k - 4}{2},$$

then $d_T(n) = k$.

By Proposition 4.6, in order to establish the truth of Conjecture 4.7, it suffices to construct a tree of order n with irregular chromatic number k. Also, observe that if T is a tree of order n with $\chi_{ir}(T) = k$, then, by Theorem 2.2, T contains at most $k(k-1)$ end-vertices and at most $k\binom{k}{2}$ vertices of degree 2. It then follows by (6) that

$$\begin{aligned} n &\leq k(k-1) + k\binom{k}{2} + [k(k-1) - 2] \\ &= \frac{k^3 + 3k^2 - 4k - 4}{2}. \end{aligned}$$

Furthermore, if T is a nontrivial tree of order

$$n = \frac{k^3 + 3k^2 - 4k - 4}{2}$$

with $\chi_{ir}(T) = k \geq 2$, then T contains exactly $k(k-1)$ end-vertices, exactly $k\binom{k}{2}$ vertices of degree 2, exactly $k(k-1) - 2$ vertices of degree 3, and no vertices of degree 4 or more.

We next establish Conjecture 4.7 for $5 \leq n \leq 100$. First, we need an additional definition. For an irregular coloring c of the path P_n: $v_1, v_2, \ldots, v_n$ of order n, define the *color sequence s_n of P_n with respect to c* as the sequence

$$s_n : c(v_1), c(v_2), \cdots, c(v_n).$$

By Proposition 4.6, the largest possible value n for which there exists a tree of order n with irregular chromatic number 3 is 19.

Proposition 4.8 *If $5 \leq n \leq 19$, then $d_T(n) = 3$.*

Proof. By Proposition 4.6, it suffices to construct a tree T_n of order n for each n with $5 \leq n \leq 14$ such that $\chi_{ir}(T_n) = 3$. It is known that $\chi_{ir}(P_n) = 3$ for $5 \leq n \leq 11$. Figure 6 shows the trees T_{18}, T_{19}, and T_{13} together with an irregular 3-coloring for each.

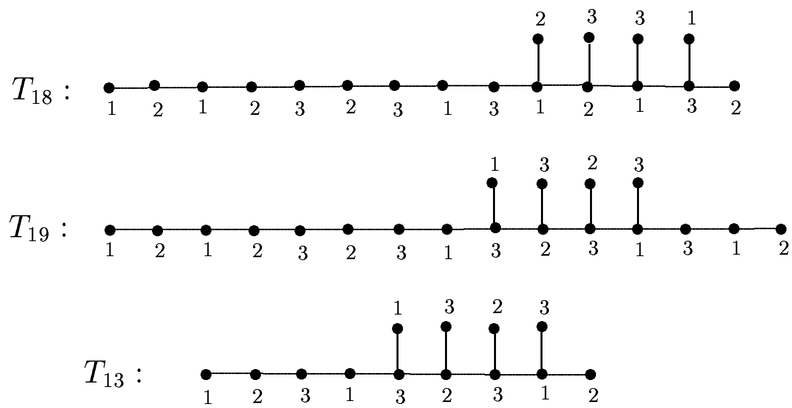

Figure 6: Irregular 3-colorings of T_{18}, T_{19}, and T_{13}

Consider the path P_{14} in the tree T_{18}, whose color sequence is

$$1, 2, 1, 2, 3, 2, 3, 1, 3, 1, 2, 1, 3, 2. \tag{7}$$

In the color sequence in (7) there are 3 subsequences of the form $ijij$, where $1 \leq i \neq j \leq 3$, namely

$$1212,\ 2323,\ 3131. \tag{8}$$

Successively replacing these subsequences $ijij$ in the sequence of (7) by ij in the order described in (8), we obtain color sequences for the paths P_{12}, P_{10}, and P_8, respectively. We can then construct T_{16}, T_{14}, and T_{12} in the same fashion as that of T_{18} as shown in Figure 6.

Similarly, consider the path P_{15} in the tree T_{19}, whose color sequence is

$$1, 2, 1, 2, 3, 2, 3, 1, 3, 2, 3, 1, 3, 1, 2. \tag{9}$$

In the color sequence in (9) there are 2 subsequences of the form $ijij$, where $1 \leq i \neq j \leq 3$, namely

$$1212,\ 2323. \tag{10}$$

Successively replacing these subsequences $ijij$ in the sequence of (9) by ij in the order described in (10), we obtain color sequences for the paths P_{13} and P_{11}, respectively. We can then construct T_{17} and T_{15} in the same fashion as that of T_{19} as shown in Figure 6. The tree T_{13} is shown in Figure 6 with an irregular 3-coloring and this completes the proof. ∎

By Proposition 4.6, the largest possible value n for which there exists a tree of order n with irregular chromatic number 4 is 46.

Proposition 4.9 *If* $20 \leq n \leq 46$, *then* $d_T(n) = 4$.

Proof. By Proposition 4.6, it suffices to construct a tree T_n of order n for each n with $20 \leq n \leq 46$ such that $\chi_{ir}(T_n) = 4$. It is known that $\chi_{ir}(P_n) = 4$ for $20 \leq n \leq 26$. Figure 7 shows the tree T_{32} together with an irregular 4-coloring. Successively deleting the end-vertices u, v, w, x, y from T_{32} produces the trees T_{31}, T_{30}, $\ldots$, T_{27}. Thus, we may assume that $33 \leq n \leq 46$.

T_{32} :

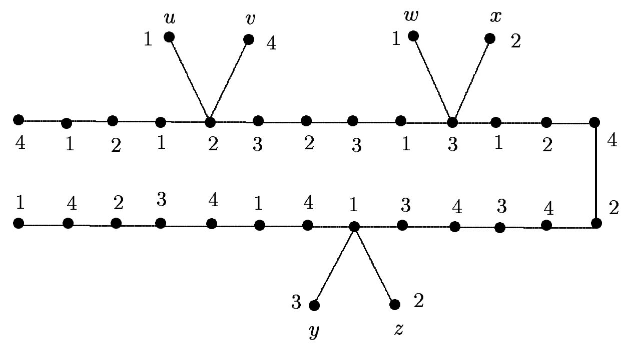

Figure 7: An irregular 4-coloring of T_{32}

First, assume that $34 \leq n \leq 46$ and n is even. The tree T_{46} is shown in Figure 8 with an irregular 4-coloring. The largest possible value n for which there exists a tree of order n with irregular chromatic number 4 is 46. Thus T_{46} contains exactly $4(4-1) = 12$ end-vertices, exactly $4\binom{4}{2} = 24$ vertices of degree 2, and exactly $4(4-1) - 2 = 10$ vertices of degree 3. Consider the $u - v$ path P_{26} in T_{46}. A color sequence for P_{26} follows:

$$4, 1, 2, 1, 2, 3, 2, 3, 1, 3, 1, 2, 4, 2, 4, 3, 4, 3, 1, 4, 1, 4, 3, 2, 4, 1. \tag{11}$$

There are 6 subsequences of the form $ijij$ in the color sequence of P_{26} in (11), namely

$$1212,\ 2323,\ 3131,\ 2424,\ 4343,\ 1414. \tag{12}$$

Successively replacing these subsequences $ijij$ by ij in the order described in (12), we obtain color sequences for T_n, where $34 \leq n \leq 44$ and n is even.

Next, assume that $33 \leq n \leq 45$ and n is odd. The tree T_{45} is shown in Figure 8 with an irregular 4-coloring. Consider the $x - y$ path P_{25} in T_{45}. There are 6 subsequences of the form $ijij$ in the color sequence of P_{25} as described in (12). Successively replacing these subsequences $ijij$ by

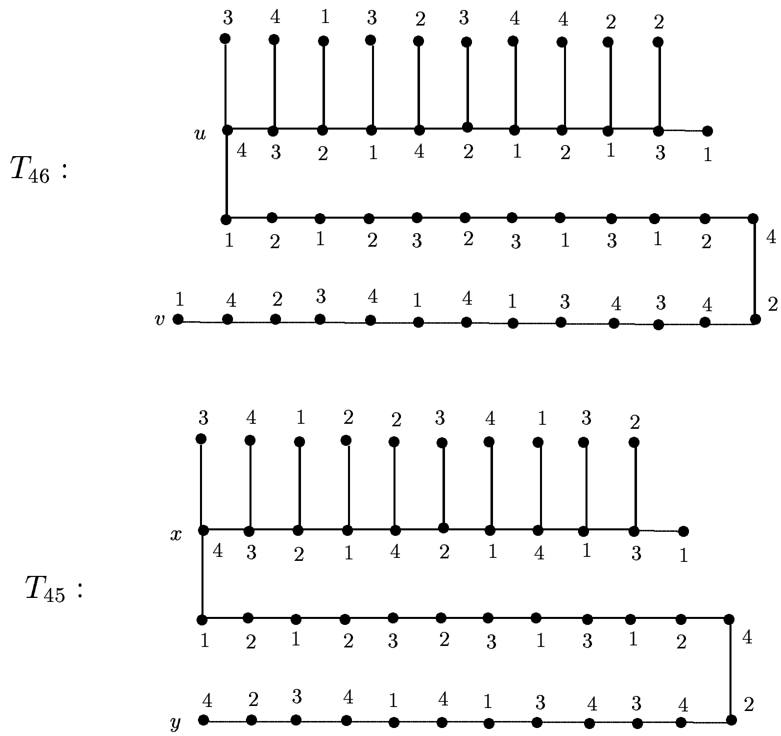

Figure 8: Irregular 4-colorings of T_{46} and T_{45}

ij in the order described in (12), we obtain color sequences for T_n, where $33 \leq n \leq 43$ and n is odd. ■

Using an argument similar to the one in the proof of Proposition 4.9, we are able to establish the following two results.

Proposition 4.10 *If* $47 \leq n \leq 88$, *then* $d_T(n) = 5$.

Proposition 4.11 *If* $89 \leq n \leq 100$, *then* $d_T(n) = 6$.

In summary, for $2 \leq n \leq 100$,

$$d_T(n) = \begin{cases} 2 & \text{if } n = 2, 4 \\ 3 & \text{if } n = 3 \text{ or } 5 \leq n \leq 14 \\ 4 & \text{if } 15 \leq n \leq 46 \\ 5 & \text{if } 47 \leq n \leq 88 \\ 6 & \text{if } 89 \leq n \leq 100. \end{cases}$$

Therefore, Conjecture 4.7 holds for $5 \leq n \leq 100$. In fact, it has also been shown that Conjecture 4.7 holds for many integers $n > 100$.

5 Vertex and Edge Removal

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. As we will see next, the addition of an edge to a graph G results in a graph whose irregular chromatic number differs from that of G by at most 2.

Proposition 5.1 *If G is a noncomplete graph and e is an edge of the complement $\overline{G}$ of G, then*

$$|\chi_{ir}(G+e) - \chi_{ir}(G)| \leq 2.$$

Proof. If $G = \overline{K}_2$, then $G + e = K_2$. Since $\chi_{ir}(\overline{K}_2) = \chi_{ir}(K_2) = 2$, the result holds for the graph of order 2. Thus, we may assume that G has at least three vertices. Suppose that $\chi_{ir}(G) = k$ and $e = uv \in E(\overline{G})$. Let c be an irregular k-coloring of G with $c(u) = i$ and $c(v) = j$, where $1 \leq i, j \leq k$. Define a coloring c^* of $G + e$ from c by

$$c^*(x) = \begin{cases} k+1 & \text{if } x = u \\ k+2 & \text{if } x = v \\ c(x) & \text{if } x \neq u, v. \end{cases}$$

We show that c^* is an irregular coloring of $G + e$. First, we make two observations.

(1) Since $c^*(u) \neq c^*(v)$, it follows that $\text{code}_{c^*}(u) \neq \text{code}_{c^*}(v)$. If $w \in V(G+e) - \{u, v\} = V(G) - \{u, v\}$, then $c^*(u) \neq c^*(w)$ and so $\text{code}_{c^*}(u) \neq \text{code}_{c^*}(w)$. Similarly, $\text{code}_{c^*}(v) \neq \text{code}_{c^*}(w)$ for $w \in V(G) - \{u, v\}$.

(2) If $w \in V(G+e) - \{u, v\} = V(G) - \{u, v\}$ and w is adjacent to neither u nor v in G (and so in $G + e$), then $\text{code}_{c^*}(w) = \text{code}_c(w)$.

Let x and y be two distinct vertices of $G+e$. If $x \in \{u, v\}$ or $y \in \{u, v\}$, then $\text{code}_{c^*}(x) \neq \text{code}_{c^*}(y)$ by (1). If $x \notin N(u) \cup N(v)$ and $y \notin N(u) \cup N(v)$, then $\text{code}_{c^*}(x) \neq \text{code}_{c^*}(y)$ by (2) and the fact that c is an irregular coloring of G. Thus we may assume (i) $x, y \in V(G) - \{u, v\}$ and (ii) $x \in N(u) \cup N(v)$ or $y \in N(u) \cup N(v)$. If exactly one of x and y is adjacent to u, say x is adjacent to u and y is not adjacent to u, then the $(k+1)$th coordinate of $\text{code}_{c^*}(x)$ is 1 and the $(k+1)$th coordinate of $\text{code}_{c^*}(y)$ is 0 and so $\text{code}_{c^*}(x) \neq \text{code}_{c^*}(y)$. Similarly, if exactly one of x and y is adjacent to v, then $\text{code}_{c^*}(x) \neq \text{code}_{c^*}(y)$. Thus if $\text{code}_{c^*}(x) = \text{code}_{c^*}(y)$, then either both x and y are adjacent to u or neither x nor y is adjacent to u. Similarly, either both x and y are adjacent to v or neither x nor y is adjacent to v.

Suppose that

$$\begin{aligned}
\text{code}_c(x) &= (x_0, x_1, x_2, \ldots, x_k),\\
\text{code}_{c^*}(x) &= (x_0^*, x_1^*, x_2^*, \ldots, x_k^*, x_{k+1}^*, x_{k+2}^*),\\
\text{code}_c(y) &= (y_0, y_1, y_2, \ldots, y_k),\\
\text{code}_{c^*}(y) &= (y_0^*, y_1^*, y_2^*, \ldots, y_k^*, y_{k+1}^*, y_{k+2}^*).
\end{aligned}$$

Since $\text{code}_c(x) \neq \text{code}_c(y)$, it follows that $x_t \neq y_t$ for some integer t with $0 \leq t \leq k$. If $t \neq i$ and $t \neq j$, then $x_t = x_t^*$ and $y_t = y_t^*$ and so $\text{code}_{c^*}(x) \neq \text{code}_{c^*}(y)$. Thus we may assume that $t = i$ or $t = j$, say $t = i$ and so $x_i \neq y_i$. If both x and y are adjacent to u, then $x_i^* = x_i - 1$ and $y_i^* = y_i - 1$; so $x_i^* \neq y_i^*$. If neither x nor y is adjacent to u, then $x_i^* = x_i$ and $y_i^* = y_i$; so $x_i^* \neq y_i^*$. Hence in either case, $\text{code}_{c^*}(x) \neq \text{code}_{c^*}(y)$. Therefore, c^* is an irregular coloring of $G + e$ using at most $k + 2$ colors and so $\chi_{ir}(G + e) \leq \chi_{ir}(G) + 2$.

To show $\chi_{ir}(G) \leq \chi_{ir}(G + e) + 2$, suppose that $\chi_{ir}(G + e) = k'$ and let c' be an irregular k'-coloring of $G + e$. We then define a coloring c of G by

$$c(x) = \begin{cases} k' + 1 & \text{if } x = u \\ k' + 2 & \text{if } x = v \\ c'(x) & \text{if } x \neq u, v. \end{cases}$$

Applying an argument similar to the one used in proving $\chi_{ir}(G + e) \leq \chi_{ir}(G) + 2$, we can show that c is an irregular coloring of G using at most $k' + 2$ colors and so $\chi_{ir}(G) \leq \chi_{ir}(G + e) + 2$. Therefore, $|\chi_{ir}(G + e) - \chi_{ir}(G)| \leq 2$. ■

By Proposition 5.1, if G is a noncomplete graph and e is an edge of $\overline{G}$, then

$$\chi_{ir}(G + e) = \chi_{ir}(G) + i, \text{ where } -2 \leq i \leq 2.$$

In fact, for each integer i with $-2 \leq i \leq 2$, there exists a graph G_i and an edge e of $\overline{G}_i$ such that $\chi_{ir}(G_i + e) = \chi_{ir}(G_i) + i$, as shown in Figure 9, where a minimum irregular coloring is also provided for each graph.

Proposition 5.2 *If G is a nontrivial graph and v is a vertex of G, then*

$$\chi_{ir}(G) - 1 \leq \chi_{ir}(G - v) \leq \chi_{ir}(G) + \deg_G v.$$

Proof. Suppose that $\chi_{ir}(G) = k$ and $N(v) = \{v_1, v_2, \ldots, v_p\}$, where then $p = \deg_G v$. Let c be an irregular k-coloring of G. Define a coloring c^* of $G - v$ by

$$c^*(x) = \begin{cases} k + i & \text{if } x = v_i \text{ where } 1 \leq i \leq p \\ c(x) & \text{if } x \notin N(v). \end{cases}$$

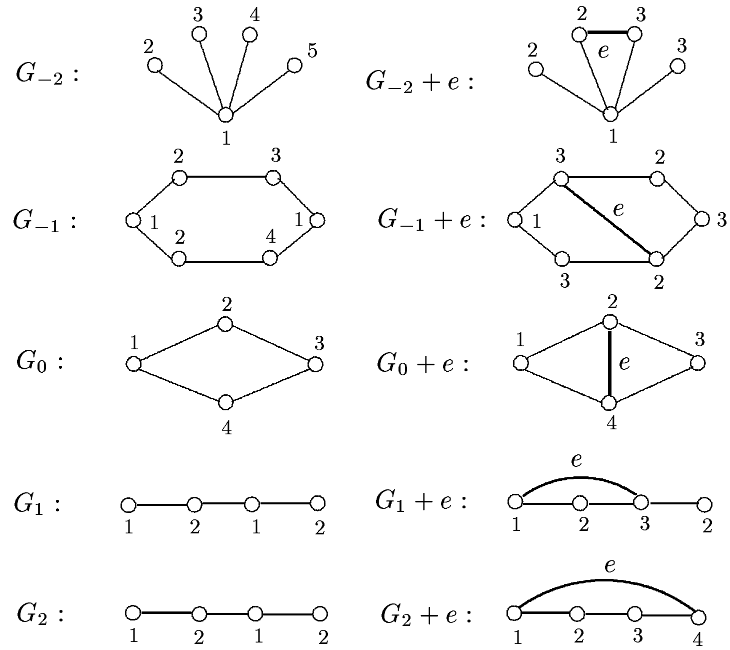

Figure 9: Graphs G_i with $\chi_{ir}(G_i+e)=\chi_{ir}(G_i)+i$ for $-2 \le i \le 2$

Thus c^* uses at most $k+p$ colors. We show that c^* is an irregular coloring of $G-v$. Let $x, y \in V(G-v)$. If $x \in N(v)$ or $y \in N(v)$, then $c^*(x) \neq c^*(y)$ and so $\text{code}_{c^*}(x) \neq \text{code}_{c^*}(y)$. Thus we may assume that $x \notin N(v)$ and $y \notin N(v)$. Suppose that

$$\begin{aligned}
\text{code}_c(x) &= (x_0, x_1, x_2, \ldots, x_k),\\
\text{code}_{c^*}(x) &= (x_0^*, x_1^*, x_2^*, \ldots, x_k^*, x_{k+1}^*, \ldots, x_{k+p}^*),\\
\text{code}_c(y) &= (y_0, y_1, y_2, \ldots, y_k),\\
\text{code}_{c^*}(y) &= (y_0^*, y_1^*, y_2^*, \ldots, y_k^*, y_{k+1}^*, \ldots, y_{k+p}^*).
\end{aligned}$$

Since $\text{code}_c(x) \neq \text{code}_c(y)$, it follows that $x_t \neq y_t$ for some integer t with $0 \le t \le k$. For each integer i, the vertex v_i is the only vertex of $G-v$ colored $k+i$. Hence if exactly one of x and y is adjacent to v_i $(1 \le i \le p)$, then $\text{code}_{c^*}(x) \neq \text{code}_{c^*}(y)$. Thus if $\text{code}_{c^*}(x) = \text{code}_{c^*}(y)$, then for each i with $1 \le i \le p$, either (1) both x and y are adjacent to v_i or (2) neither x nor y is adjacent to v_i. Hence $N(x) \cap N(v) = N(y) \cap N(v)$. Let $q = |N(x) \cap N(v)|$. Then $x_t^* = x_t - q$ and $y_t^* = y_t - q$, implying that $\text{code}_{c^*}(x) \neq \text{code}_{c^*}(y)$. Therefore, c^* is an irregular coloring of $G-v$ using at most $k+p$ colors. Thus $\chi_{ir}(G-v) \le \chi_{ir}(G) + \deg_G v$.

Next we show that $\chi_{ir}(G)-1 \leq \chi_{ir}(G-v)$. Suppose that $\chi_{ir}(G-v) = k'$ and c' is an irregular k'-coloring of $G - v$. We now add v to $G - v$ and join v to each of the vertices $v_1, v_2, \ldots, v_p$ of $G - v$. Define a coloring c of G from c' by

$$c(x) = \begin{cases} k'+1 & \text{if } x = v \\ c'(x) & \text{if } x \neq v. \end{cases}$$

Then c uses $k' + 1$ colors. We now show that c is an irregular coloring of G. Let $x, y \in V(G)$. If $x = v$ or $y = v$, then $c(x) \neq c(y)$ and so $\text{code}_c(x) \neq \text{code}_c(y)$. Thus we may assume that $x, y \in V(G) - \{v\}$. Since c' is an irregular coloring of $G - v$, it follows that $\text{code}_{c'}(x) \neq \text{code}_{c'}(y)$. If x is adjacent to v, then the first $(k' + 1)$ coordinates of $\text{code}_c(x)$ are same as those in $\text{code}_{c'}(x)$ and the $(k' + 2)$th coordinate of $\text{code}_c(x)$ is 1. If x is not adjacent to v, then the first $(k' + 1)$ coordinates of $\text{code}_c(x)$ are same as those in $\text{code}_{c'}(x)$ and the $(k' + 2)$th coordinate of $\text{code}_c(x)$ is 0. The same can be said about the coordinates of $\text{code}_c(y)$. If exactly one of x and y is adjacent to v, then $\text{code}_c(x) \neq \text{code}_c(y)$ since their $(k' + 2)$th coordinates are different. If both x and y are adjacent to v or neither x nor y is adjacent to v, then $\text{code}_c(x) \neq \text{code}_c(y)$ since their first $(k' + 1)$ coordinates (namely, those in $\text{code}_{c'}(x)$ and $\text{code}_{c'}(y)$) are different. Thus c is an irregular coloring of G using $k'+1$ colors and so $\chi_{ir}(G) \leq \chi_{ir}(G-v)+1$. Therefore, $\chi_{ir}(G - v) \geq \chi_{ir}(G) - 1$. ∎

Next we show that both upper and lower bounds in Proposition 5.2 are sharp, beginning with the upper bound. In fact, for each integer $p \geq 1$, there exists a graph G_p having a vertex v of degree p such that $\chi_{ir}(G_p-v) = \chi_{ir}(G_p) + p$, as we show next. For $p = 1$, let $G_1 = P_4$ and let v be an end-vertex of G_1. Then $G_1 - v = P_3$. Since $\chi_{ir}(P_4) = 2$ and $\chi_{ir}(P_3) = 3$, it follows that $\chi_{ir}(G_1 - v) = \chi_{ir}(G_1) + 1$. For $p \geq 2$, let $F_p = K_{3,2,2,\ldots,2}$ be the complete p-partite graph of order $3 + 2(p - 1) = 2p + 1$, whose partite sets are $V_1, V_2, \ldots, V_p$, where $V_1 = \{v_{1,1}, v_{1,2}, v_{1,3}\}$ and $V_i = \{v_{i,1}, v_{i,2}\}$ for $2 \leq i \leq p$. The graph G_p is then obtained from F_p by adding a new vertex v and joining v to each vertex $v_{i,1}$ for $1 \leq i \leq p$. Thus $\deg_{G_p} v = p$. Since $G_p - v = F_p = K_{3,2,2,\ldots,2}$, it follows that $\chi_{ir}(G_p - v) = \chi_{ir}(K_{3,2,2,\ldots,2}) = 2p + 1$. Next, we show that $\chi_{ir}(G_p) = p + 1$. Let

$$S = \{v\} \cup \{v_{i,1} : 1 \leq i \leq p\}.$$

Since $\langle S \rangle = K_{p+1}$, it follows that G_p contains a complete graph K_{p+1} as a subgraph and so $\chi_{ir}(G_p) \geq p + 1$. To show that $\chi_{ir}(G_p) \leq p + 1$, we define a $(p + 1)$-coloring c of G_p by

$$c(x) = \begin{cases} p+1 & \text{if } x = v \text{ or } x = v_{1,3} \\ i & \text{if } x \in V_i \text{ for } 1 \leq i \leq p \text{ and } x \neq v_{1,3}. \end{cases}$$

Next, we show that c is an irregular $(p+1)$-coloring of G_p. If $x, y \in V(G_p)$ such that $c(x) = c(y)$, then either $\{x, y\} = \{v, v_{1,3}\}$ or $\{x, y\} = \{v_{i,1}, v_{i,2}\}$ for some i with $1 \leq i \leq p$. Suppose first that $\{x, y\} = \{v, v_{1,3}\}$, say $x = v$ and $y = v_{1,3}$. Then x is adjacent to a vertex colored 1 (namely $v_{1,1}$) but y is not adjacent to any vertex colored 1, implying that $\text{code}_c(x) \neq \text{code}_c(y)$. Next, suppose that $\{x, y\} = \{v_{i,1}, v_{i,2}\}$, say $x = v_{i,1}$ and $y = v_{i,2}$. If $i = 1$, then x is adjacent to a vertex colored $p+1$ (namely v) but y is not adjacent to any vertex colored $p+1$; while if $2 \leq i \leq p$, then x is adjacent to two vertices colored $p+1$ but y is adjacent to exactly one vertex colored $p+1$. In either case, $\text{code}_c(x) \neq \text{code}_c(y)$. Thus c is an irregular $(p+1)$-coloring of G_p and so $\chi_{ir}(G_p) \leq p+1$. Hence $\chi_{ir}(G_p) = p+1$. Therefore, $\chi_{ir}(G_p - v) = \chi_{ir}(G_p) + p$ for all $p \geq 2$. For $p = 2, 3$, the graphs G_p and $G_p - v$ are shown in Figure 10, where a minimum irregular coloring is provided for each graph.

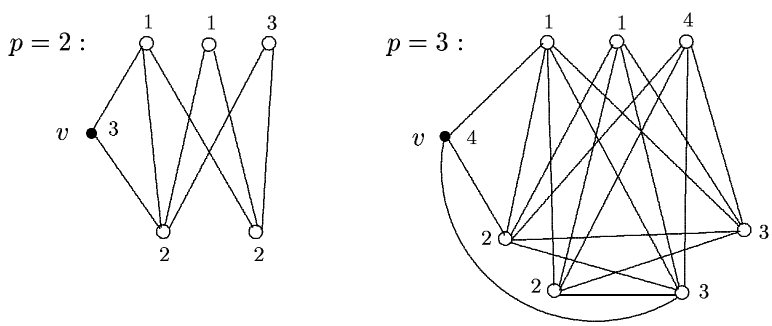

Figure 10: Graphs G_p and $G_p - v$ for $p = 2, 3$

To verify the lower bound in Proposition 5.2 is sharp, let $G = K_{1,n-1}$ where $n \geq 3$. Then $\chi_{ir}(G) = n$. For a vertex v of G, either $G - v = K_{1,n-2}$ or $G - v = \overline{K}_{n-1}$. Since $\chi_{ir}(K_{1,n-2}) = \chi_{ir}(\overline{K}_{n-1}) = n - 1$, it follows that $\chi_{ir}(G - v) = \chi_{ir}(G) - 1$ for every vertex v in G.

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