Irregular Colorings of Graphs

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Abstract

Let G be a connected graph and let $c : V(G) \to \{1, 2, \ldots, k\}$ be a proper coloring of the vertices of G for some positive integer k. The color code of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple $\text{code}(v) = (a_0, a_1, \ldots, a_k)$ where a_0 is the color assigned to v and for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored i. The coloring c is irregular if distinct vertices have distinct color codes and the irregular chromatic number $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k-coloring. Characterizations of connected graphs of order n having irregular chromatic numbers 2 or n are established. For a pair k, n of integers with $2 \leq k \leq n$, it is shown that there exists a connected graph of order n having irregular chromatic number k if and only if $(k, n) \neq (2, n)$ for some odd integer n. Irregular chromatic numbers of cycles are investigated.

Key Words: irregular coloring, irregular chromatic number.
AMS Subject Classification: 05C15.

1 Introduction

A problem in graph theory that has received increased attention during the past 35 years concerns studying methods to distinguish the vertices of a connected graph. One of the earlier methods is due to Sumner [15] and Entringer and Gassman [9]. They studied graphs G for which the equality of the open neighborhoods of every two vertices of G implies that the vertices are the same. In this case, the vertices of G are uniquely determined by their open neighborhoods.

Erwin and Harary [10] introduced the idea of selecting a subset S of the vertex set of a graph G such that the subgroup of $\text{Aut}(G)$ that fixes every vertex of S is the identity group. Albertson and Collins [3] and Harary [11] introduced the notion of coloring the vertices of G in such a way that the subgroup of color-preserving automorphisms of $\text{Aut}(G)$ is the identity group. In these ways, the vertices of a graph G can be distinguished from one another with the aid of certain automorphisms of G.

Another way to distinguish the vertices of a connected graph G from one another was introduced by Harary and Melter [12] and Slater [14]. In this method, an ordered set W of vertices of G, say $W=\{w_1, w_2, \ldots, w_k\}$, is found and each vertex v is assigned the ordered k-tuple $c_W(v)= (a_1, a_2, \ldots, a_k)$, where $a_i = d(v, w_i)$ is the distance between v and w_i for $1 \leq i \leq k$. The ordered k-tuple $c_W(v)$ is sometimes called the *distance code* of v. If distinct vertices of G have distinct distance codes, then the vertices of G are distinguishable.

Harary and Plantholt [13] introduced yet another way to distinguish the vertices of a graph G by assigning colors to the edges of G in such a way that for every two vertices of G, one of the vertices is incident with an edge assigned one of these colors that the other vertex is not. They referred to the minimum number of colors needed to accomplish this as the *point-distinguishing chromatic index* of G.

There is still another manner in which differences among the vertices of a connected graph G can be detected. Let G be a connected graph of order $n \geq 3$ and let $c : E(G) \to \{1, 2, \ldots, k\}$ be a coloring of the edges of G for some positive integer k (where adjacent edges may be colored the same). The *color code* of a vertex v of G with respect to a k-coloring c of the edges of G is the ordered k-tuple $(a_1, a_2, \ldots, a_k)$ where a_i is the number of edges incident with v that are colored i for $1 \leq i \leq k$. The edge-coloring c is *detectable* if distinct vertices have distinct color codes. The minimum positive integer k for which G has a detectable k-coloring is the *detection number* of G. The concept of detectable colorings was studied in [1, 2, 4, 5, 6].

In [7] a method was introduced to recognize the vertices of a graph. Let G be a graph and let $c : V(G) \to \{1, 2, \ldots, k\}$ be a coloring of the vertices of G for some positive integer k (where adjacent vertices may be colored the same). The *color code* of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple $\text{code}_c(v) = (a_0, a_1, \ldots, a_k)$ where a_0 is the color assigned to v and for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored i. The coloring c is called *recognizable* if distinct vertices have distinct color codes and the *recognition number* of G is the minimum positive integer k for which G has a recognizable k-coloring.

In this work, we introduce a method that combines a number of the features of the methods mentioned above. We refer to the book [8] for graph theory notation and terminology not described in this paper.

2 Basic Definitions and Preliminary Results

A (proper) *coloring* of a graph G is a function $c : V(G) \to \mathbf{N}$ having the property that $c(u) \neq c(v)$ for every pair u, v of adjacent vertices of G. A *k-coloring* of G uses k colors. The *chromatic number* $\chi(G)$ of G is the minimum integer k for which G admits a k-coloring.

Let G be a graph and let $c : V(G) \to \{1, 2, \ldots, k\}$ be a proper coloring of the vertices of G for some positive integer k. The *color code* of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple

$$\text{code}(v) = (a_0, a_1, \ldots, a_k) \quad (\text{or simply, code}(v) = a_0 a_1 a_2 \cdots a_k),$$

where a_0 is the color assigned to v (that is, $a_0 = c(v)$) and for $1 \le i \le k$, a_i is the number of vertices adjacent to v that are colored i. Therefore, if $a_0 = i$, then $a_i = 0$ for $1 \le i \le k$ and $\sum_{i=1}^{k} a_i = \deg_G v$. The coloring c is called *irregular* if distinct vertices have distinct color codes and the *irregular chromatic number* $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k-coloring. An irregular k-coloring with $\chi_{ir}(G) = k$ is a *minimum irregular coloring.* Since every irregular coloring of a graph G is a coloring of G, it follows that

$$\chi(G) \le \chi_{ir}(G). \tag{1}$$

To illustrate this concept, consider the Petersen graph P of Figure 1. Since $\chi(P) = 3$, it follows by (1) that $\chi_{ir}(P) \ge 3$. A 4-coloring of the Petersen graph is given in Figure 1 along with the corresponding color codes of its vertices. Since distinct vertices have distinct codes, this coloring is irregular and so $\chi_{ir}(P) \le 4$. Therefore, $\chi_{ir}(P) = 3$ or $\chi_{ir}(P) = 4$. We show that $\chi_{ir}(P) = 4$. Assume, to the contrary, that $\chi_{ir}(P) = 3$. Let c be an irregular 3-coloring of P. Let u and v be two vertices of P with $c(u) = c(v)$. We may assume that $c(u) = c(v) = 1$. Since the diameter of P is 2 and u and v are nonadjacent, there is a path u, w, v in P. This implies that at most one of u and v is adjacent to three vertices having the same color. That is, no two vertices colored 1 can have the two color codes 1030 and 1003. Thus at most three vertices of P can be colored 1 and, in general, at most three vertices of P can be assigned the same color, contradicting our assumption that $\chi_{ir}(P) = 3$.

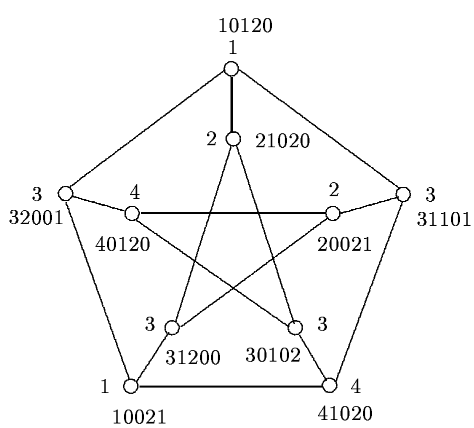

Figure 1: An irregular 4-coloring of the Petersen graph P

There are some observations that will be useful to us.

Observation 2.1 *Let c be a coloring of the vertices of a graph G. If u and v are two vertices of G with $c(u) \ne c(v)$, then* $\text{code}(u) \ne \text{code}(v)$.

Observation 2.2 *Let c be a coloring of the vertices of a graph G. If u and v are two vertices of G with $\deg_G u \neq \deg_G v$, then $\text{code}(u) \neq \text{code}(v)$.*

By Observations 2.1 and 2.2, to show that a coloring of a graph G is irregular, it is necessary and sufficient to show that every two vertices of same degree and same color have distinct codes.

The *neighborhood* of a vertex u in a graph G is $N(u) = \{v \in V(G) : uv \in E(G)\}$.

Observation 2.3 *Let c be an irregular coloring of a graph G. If u and v are distinct vertices of G with $N(u) = N(v)$, then $c(u) \neq c(v)$.*

The following result, dealing with combinations with repetition, is well-known in discrete mathematics.

Theorem A *Let A be a set containing ℓ different kinds of elements, where there are at least r elements of each kind. The number of different selections of r elements from A is $\binom{r+\ell-1}{r}$.*

We have seen that if c is an irregular k-coloring of a graph G and $v \in V(G)$ such that $\text{code}(v) = (i, a_1, \ldots, a_k)$ for some i with $1 \leq i \leq k$, then $a_i = 0$ and the sum of the remaining $k-1$ coordinates $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots a_k$, is the degree of v. Therefore, by Theorem A, we have the following.

Theorem 2.4 *Let c be an irregular k-coloring of the vertices of a graph G. The number of different possible color codes of the vertices of degree r in G is*

$$k\binom{r+(k-1)-1}{r} = k\binom{r+k-2}{r}.$$

The following result is a consequence of Theorem 2.4.

Corollary 2.5 *If c is an irregular k-coloring of a nontrivial connected graph G, then G contains at most $k\binom{r+k-2}{r}$ vertices of degree r.*

As a consequence of Corollary 2.5, if c is an irregular 2-coloring of a graph G, then G has at most two vertices of the same degree. Thus if G contains three vertices of the same degree, then $\chi_{ir}(G) \geq 3$. This observation yields the following.

Corollary 2.6 *If G is an r-regular graph for $r \geq 2$, then $\chi_{ir}(G) \geq 3$.*

Thus if c is an irregular 3-coloring of a graph G, then G contains at most $3(r+1)$ vertices of each degree r. In particular, G contains at most 9 vertices of degree 2 and so the largest possible order of a cycle with irregular chromatic number 3 is 9. Since C_9 has an irregular 3-coloring (see Figure 2), it follows that $\chi_{ir}(C_9) = 3$.

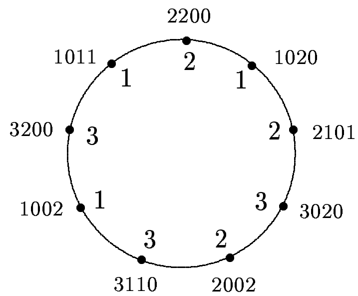

Figure 2: A minimum irregular coloring of C_9

Furthermore, if c is an irregular 3-coloring of a cubic graph G (a 3-regular graph), then G contains at most 12 vertices, that is, the largest possible order of a cubic graph with irregular chromatic number 3 is 12. The cubic graph G of Figure 3 has order 12 and $\chi_{ir}(G) = 3$.

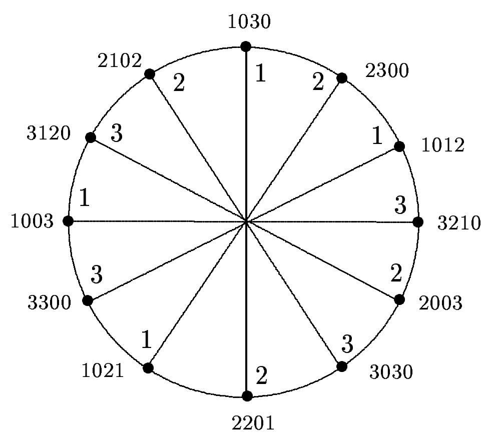

Figure 3: An irregular 3-coloring of a cubic graph of order 12

3 Irregular Colorings of Cycles

We have seen that $\chi_{ir}(P_9) = 3$. We now determine the irregular chromatic number of other cycles. The following are consequences of Corollary 2.5.

Corollary 3.1 *If* $\chi_{ir}(C_n) = k$, *where* $n \geq 3$, *then* $n \leq k\binom{k}{2} = \frac{k^2(k-1)}{2}$.

Corollary 3.2 *Let $k \geq 3$ be an integer. Then $\chi_{ir}(C_n) \geq k$ for all integers n such that*

$$\frac{(k-1)^2(k-2)+2}{2} \leq n \leq \frac{k^2(k-1)}{2}.$$

Each of the colorings in Figure 4 is a minimum irregular coloring. Thus for $3 \leq n \leq 9$ $\chi_{ir}(C_n) = 4$ if n is even and $\chi_{ir}(C_n) = 3$ if n is odd.

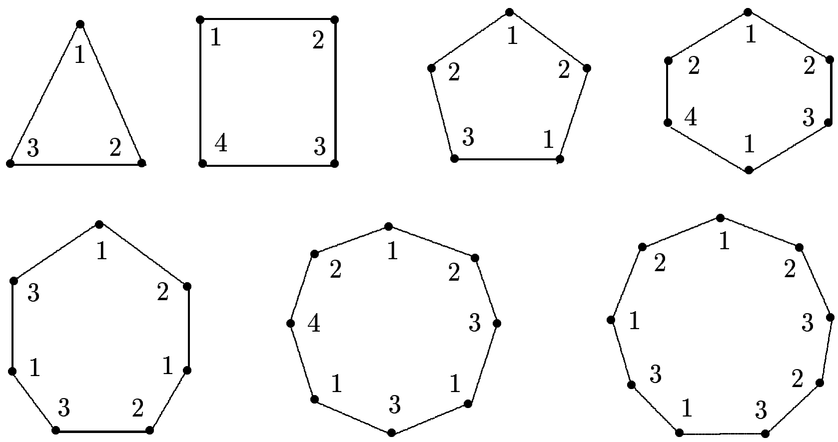

Figure 4: Minimum irregular colorings of C_n for $3 \leq n \leq 9$

The largest possible value of n for which $\chi_{ir}(C_n) = 4$ is 24. In fact, $\chi_{ir}(C_{24}) = 4$ as the irregular 4-coloring of C_{24} shows in Figure 5.

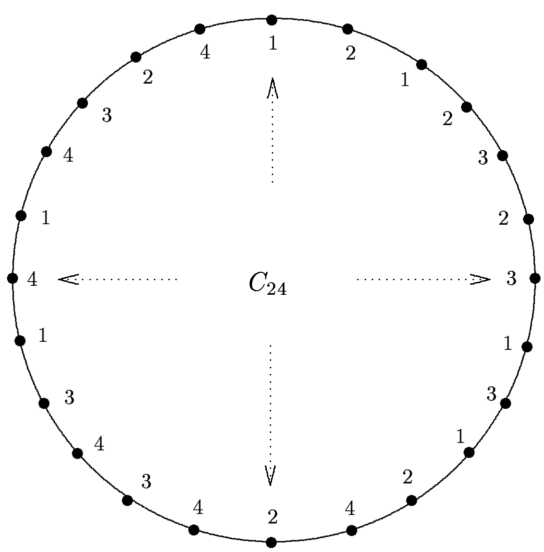

Figure 5: An irregular 4-coloring of C_{24}

The largest possible value of n for which $\chi_{ir}(C_n) = 5$ is 50. In fact, $\chi_{ir}(C_{50}) = 5$ as the irregular 5-coloring of C_{50} shows in Figure 6.

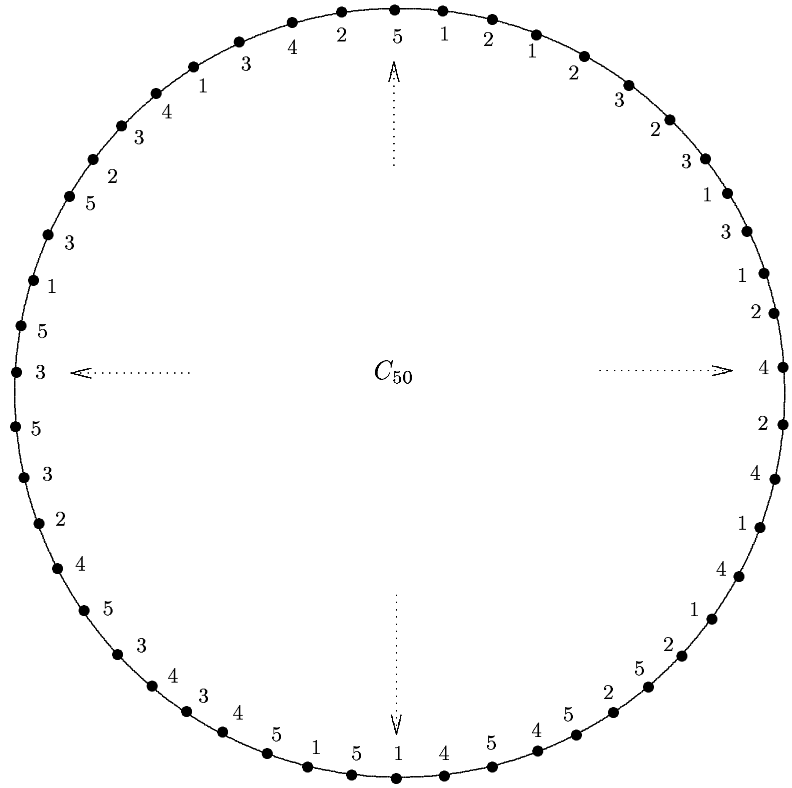

Figure 6: An irregular 5-coloring of C_{50}

We describe how the irregular 4-coloring of C_{24} in Figure 5 was constructed. (The irregular 5-coloring of C_{50} shown in Figure 6 was constructed by a similar approach.) Let $S = \{1, 2, 3, 4\}$. With the aid of a so-called deBruijn digraph, a cyclic sequence

$$s: \ a_1, a_2, \ldots, a_{64}, a_{65} = a_1, a_{66} = a_2$$

of length 64 can be constructed whose terms are the elements of S and having the property that the 3-term subsequences a_i, a_{i+1}, a_{i+2} $(1 \leq i \leq 64)$ are all 64 3-permutations (with repetition) of the elements of S. This deBruijn digraph D has order 16 and $V(D)$ consists of the 2-permutations (with repetition) of the elements of S. An arc e of D joins two vertices ab and cd, where $a, b, c, d \in S$, and is directed from ab to cd if and only if $b = c$, and e is labeled abd. For example, 12 is joined to 23 (resulting in an arc labeled 123), but 23 is not joined to 12; while 12 is joined to 21 (resulting in an arc labeled 121) and 21 is joined to 12 (resulting in an arc labeled 212). Also, 11 is joined to 11 by a directed loop labeled 111. The resulting digraph D is connected and every vertex of D has outdegree 4 and indegree 4. This implies that D is Eulerian and so contains an Eulerian circuit whose 64 arcs can therefore be listed cyclically as e_1, e_2, $\ldots$, e_{64}, $e_{65} = e_1$ so that the labels of these arcs are the 3-permutations of the elements of S and such that if abc is the label of e_i $(1 \leq i \leq 64)$, then bcd is the label of e_{i+1} for some $d \in S$.

In this case, we are interested in a particular subdigraph of the deBruijn digraph D. Let $U = \{ii:\ 1 \le i \le 4\}$. First let D^* be the subdigraph induced by $V(D) - U$, that is, $D^* = \langle V(D) - U \rangle$. Then the order of D^* is 12 and $V(D^*)$ consists of the 2-permutations ij of distinct elements of S. We are seeking a cyclic sequence

$$s':\ b_1, b_2, \ldots, b_{23}, b_{24} = b_1, b_{25} = b_2$$

of length 24 whose terms are the elements of S and having the property that the 3-term subsequences b_i, b_{i+1}, b_{i+2} $(1 \le i \le 24)$ are all 24 3-permutations abc of the elements of S such that (1) $b \ne a, c$ and (2) exactly one of a, b, c and c, b, a occurs among the 3-term subsequences of s'. In order to construct such a sequence s', we seek a spanning Eulerian subdigraph D' of D^* such that for each 3-permutation abc of the elements of S for which $b \ne a, c$, exactly one arc of D' is labeled abc or cba. For example, D' must contain (i) both of the arcs labeled 121 and 212 and (ii) exactly one of the arcs labeled 123 and 321. The irregular 4-coloring of C_{24} in Figure 5 was constructed by finding an Eulerian subdigraph D' of D^* and an Eulerian circuit C' of D'.

Even though $\chi_{ir}(C_{24}) = 4$ and $\chi_{ir}(C_{50}) = 5$, it turns out that $\chi_{ir}(C_{23}) > 4$ and $\chi_{ir}(C_{49}) > 5$. In general, we have the following.

Theorem 3.3 *Let $k \ge 3$. If $n = \frac{k^2(k-1)}{2}$, then $\chi_{ir}(C_{n-1}) \ge k + 1$.*

Proof. Assume, to the contrary, that $\chi_{ir}(C_{n-1}) \le k$. Then there exists an irregular k-coloring c of C_{n-1}. Since $\frac{k^2(k-1)}{2}$ is the largest possible value of n for which $\chi_{ir}(C_n) = k$, any irregular k-coloring of C_n, should it exist, must result in exactly $n = \frac{k^2(k-1)}{2}$ distinct color codes for the n vertices of C_n, and so exactly $\frac{k^2(k-1)}{2k} = \frac{k(k-1)}{2}$ vertices of C_n are colored i for each i with $1 \le i \le k$. In the irregular k-coloring c of C_{n-1}, there must be exactly $n - 1 = \frac{k^2(k-1)}{2} - 1$ distinct color codes, implying that exactly one of the n distinct color codes for C_n is not used. Therefore, c assigns one of the k colors to exactly $\frac{k(k-1)}{2} - 1$ vertices of C_{n-1} and assigns each of the remaining $k - 1$ colors to exactly $\frac{k(k-1)}{2}$ vertices of C_{n-1}. We may assume, without loss of generality, that c assigns color 1 to exactly $\frac{k(k-1)}{2} - 1$ vertices of C_{n-1}. Let $S = \{v \in V(C_{n-1}):\ c(v) = 1\}$. Then $|S| = \frac{k(k-1)}{2} - 1$. Let

$$N(S) = \bigcup\nolimits_{v \in S} N(v).$$

Thus $2 \le c(x) \le k$ for each $x \in N(S)$. Since all possible color codes must be used for vertices colored i for each i with $2 \le i \le k$, there exists exactly one vertex v_i colored i that has two neighbors u_i and w_i colored 1. However then $v_i \in N(u_i) \cap N(w_i)$. This says that

$$|N(S)| = 2|S| - (k-1) = k(k-1) - 2 - (k-1) = (k-1)^2 - 2. \tag{2}$$

Since each vertex in $N(S)$ is assigned one of the colors $2, 3, \ldots, k$, the color code of each vertex in $N(S)$ has the form $(i, a_1, a_2, \cdots, a_k)$, where $i \in \{2, 3, \ldots, k\}$, $a_1 \in \{1, 2\}$, and

$a_i = 0$. For each i with $2 \le i \le k$, exactly one such color code has $a_1 = 2$ and exactly $k-2$ color codes have $a_1 = 1$, for a total of $k-1$ distinct color codes for each i. Thus there are exactly $(k-1)^2$ distinct color codes for the vertices of $N(S)$. Since all of these color codes are used for the vertices of $N(S)$, it follows that $|N(S)| = (k-1)^2$, which contradicts (2). ■

Corollary 3.4 *Let $k \ge 3$ and $n = \frac{k^2(k-1)}{2}$. If $\chi_{ir}(C_n) = k$, then $\chi_{ir}(C_{n-1}) = k+1$.*

Proof. Let c be an irregular k-coloring of C_n. Since $\frac{k^2(k-1)}{2}$ is the largest possible value of n for which $\chi_{ir}(C_n) = k$, it follows that c results in exactly $\frac{k^2(k-1)}{2}$ distinct color codes for the n vertices of C_n. Thus there exist four consecutive vertices s, u, v, t on C_n such that $c(s) = c(v) = 1$, $c(u) = 2$, and $c(t) = k \ge 3$. The cycle C_{n-1} can be constructed from C_n by identifying the vertices u and v, resulting in a vertex w. Then the $(k+1)$-coloring c' of C_{n-1} defined by $c'(x) = k+1$ if $x = w$ and $c'(x) = c(x)$ if $x \neq w$ is irregular and so $\chi_{ir}(C_{n-1}) \le k+1$. By Theorem 3.3, $\chi_{ir}(C_{n-1}) = k+1$. ■

Next, we determine the irregular chromatic number of cycles of order n for $10 \le n \le 50$. For an irregular coloring c of the cycle $C_n : v_1, v_2, \ldots, v_n, v_1$ of order n, define the *color sequence* of C_n with respect to c as the cyclic sequence

$$s_n : c(v_1), c(v_2), \cdots, c(v_n), c(v_1).$$

For example, for the irregular 4-coloring of C_{21} of Figure 7, the color sequence is

$$s_{21} : 1, 2, 1, 2, 3, 2, 3, 4, 3, 4, 1, 4, 1, 3, 1, 3, 4, 2, 4, 2, 3, 1 \quad (3)$$

Therefore, an irregular coloring of a cycle can be represented by its color sequence.

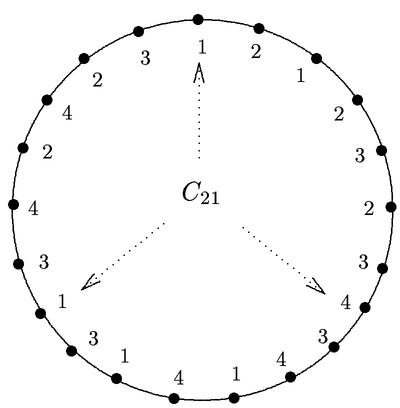

Figure 7: An irregular 4-coloring of C_{21}

Proposition 3.5 *For $10 \le n \le 24$, $\chi_{ir}(C_n) = 4$ if $n \neq 23$ and $\chi_{ir}(C_{23}) = 5$.*

Proof. The color sequence of C_{24} with respect to the irregular 4-coloring shown in Figure 5 is

$$s_{24} : 1, 2, 1, 2, 3, 2, 3, 1, 3, 1, 3, 2, 4, 2, 4, 3, 4, 3, 1, 4, 1, 4, 3, 2, 4, 1. \tag{4}$$

We first consider $\chi_{ir}(C_n)$ for each even integer n with $10 \le n \le 22$. In the color sequence of C_{24} shown in (4) there are 6 subsequences of the form i, j, i, j (or more simply $ijij$), where $1 \le i \ne j \le 4$, namely

$$1212,\ 2323,\ 3131,\ 2424,\ 4343,\ 1414. \tag{5}$$

Successively replacing these subsequences $ijij$ in s_{24} by ij in the order described in (5), we obtain color sequences for C_n, where $12 \le n \le 22$ and n is even. These color sequences are

$$\begin{array}{ll}
s_{22}: & 1, 2, 3, 2, 3, 1, 3, 1, 3, 2, 4, 2, 4, 3, 4, 3, 1, 4, 1, 4, 3, 2, 4, 1 \\
s_{20}: & 1, 2, 3, 1, 3, 1, 3, 2, 4, 2, 4, 3, 4, 3, 1, 4, 1, 4, 3, 2, 4, 1 \\
s_{18}: & 1, 2, 3, 1, 3, 2, 4, 2, 4, 3, 4, 3, 1, 4, 1, 4, 3, 2, 4, 1 \\
s_{16}: & 1, 2, 3, 1, 3, 2, 4, 3, 4, 3, 1, 4, 1, 4, 3, 2, 4, 1 \\
s_{14}: & 1, 2, 3, 1, 3, 2, 4, 3, 1, 4, 1, 4, 3, 2, 4, 1 \\
s_{12}: & 1, 2, 3, 1, 3, 2, 4, 3, 1, 4, 3, 2, 4, 1.
\end{array}$$

An irregular 4-coloring of C_{10} can be obtained from an irregular 3-coloring of C_9 by assigning color 4 to the vertex v_{10}. Therefore, $\chi_{ir}(C_n) = 4$ for $10 \le n \le 22$ and n is even.

Next, we consider $\chi_{ir}(C_n)$ for each odd integer n with $11 \le n \le 23$. By Corollary 3.4, $\chi_{ir}(C_{23}) = 5$ and $\chi_{ir}(C_{21}) = 4$ by the irregular 4-coloring of C_{21} in Figure 7. Deleting the subsequence 43241 from the color sequence of C_{24} in (4), we obtain a color sequence for C_{19} as follows:

$$s_{19} : 1, 2, 1, 2, 3, 2, 3, 1, 3, 1, 3, 2, 4, 2, 4, 3, 4, 3, 1, 4, 1. \tag{6}$$

There are 4 subsequences of the form $ijij$ in the color sequence of C_{19} in (6), namely

$$2323,\ 3131,\ 2424,\ 4343. \tag{7}$$

Successively replacing these subsequences $ijij$ by ij in the order described in (7), we obtain color sequences for C_{17}, C_{15}, C_{13} and C_{13}, namely

$$\begin{array}{ll}
s_{17}: & 2, 1, 2, 3, 1, 3, 1, 3, 2, 4, 2, 4, 3, 4, 3, 1, 4, 1, 2 \\
s_{15}: & 2, 1, 2, 3, 1, 3, 2, 4, 2, 4, 3, 4, 3, 1, 4, 1, 2 \\
s_{13}: & 2, 1, 2, 3, 1, 3, 2, 4, 3, 4, 3, 1, 4, 1, 2 \\
s_{11}: & 2, 1, 2, 3, 1, 3, 2, 4, 3, 1, 4, 1, 2.
\end{array}$$

Therefore, $\chi_{ir}(C_n) = 4$ for $11 \le n \le 21$ and n is odd. ■

Proposition 3.6 *For* $25 \le n \le 50$, $\chi_{ir}(C_n) = 5$ *if* $n \ne 49$ *and* $\chi_{ir}(C_{49}) = 6$.

Proof. The color sequence of C_{50} with respect to the irregular 5-coloring in Figure 6 is

$$\begin{aligned} s_{50} : &1, 2, 1, 2, 3, 2, 3, 1, 3, 1, 3, 2, 4, 2, 4, 1, 4, 1, 4, 2, 5, 2, 5, 4, 5, 4, 5, \\ &1, 5, 1, 5, 4, 3, 4, 3, 5, 4, 2, 3, 5, 3, 5, 1, 3, 5, 2, 3, 4, 1, 3, 4, 2, 5, 1. \end{aligned} \tag{8}$$

We first consider $\chi_{ir}(C_n)$ for each even integer n with $26 \leq n \leq 48$. In the color sequence of C_{50} shown in (8) there are 10 subsequences of the form or $ijij$, where $1 \leq i \neq j \leq 5$, namely

$$1212,\ 2323,\ 3131,\ 2424,\ 4343,\ 1414, 2525, 5454, 1515, 4343, 3535. \tag{9}$$

Successively replacing these subsequences $ijij$ by ij in the order described in (9), we obtain color sequences for C_n, where $30 \leq n \leq 48$ and n is even. Furthermore, a color sequence for each of C_{26} and C_{28} is shown as follows.

$$\begin{aligned} s_{26} : &\quad 1, 5, 2, 1, 2, 5, 3, 2, 3, 1, 3, 1, 3, 2, 4, 2, 4, 3, 4, 3, 1, 4, 1, 4, 3, 2, 4, 1 \\ s_{28} : &\quad 1, 5, 2, 1, 2, 5, 3, 2, 3, 5, 1, 3, 1, 3, 5, 2, 4, 2, 4, 3, 4, 3, 1, 4, 1, 4, 3, 2, 4, 1 \end{aligned}$$

Therefore, $\chi_{ir}(C_n) = 5$ for $26 \leq n \leq 48$ and n is even.

Next, we consider $\chi_{ir}(C_n)$ for each odd integer n with $25 \leq n \leq 49$. By Corollary 3.4, $\chi_{ir}(C_{49}) = 6$. Deleting the subsequence 134 from the color sequence of C_{50} in (8), we obtain a color sequence for C_{47} as follows:

$$\begin{aligned} s_{47} : &1, 2, 1, 2, 3, 2, 3, 1, 3, 1, 3, 2, 4, 2, 4, 1, 4, 1, 4, 2, 5, 2, 5, 4, 5, 4, 5, \\ &1, 5, 1, 5, 4, 3, 4, 3, 5, 4, 2, 3, 5, 3, 5, 1, 3, 5, 2, 3, 4, 2, 5, 1. \end{aligned} \tag{10}$$

There are 10 subsequences of the form $ijij$ in the color sequence of C_{47} in (9). Successively replacing these subsequences $ijij$ by ij in the order described in (9), we obtain color sequences for C_n, where $27 \leq n \leq 45$ and n is odd. An irregular 5-coloring of C_{25} can be obtained from an irregular 4-coloring of C_{24} by assigning color 5 to the vertex v_{25}. Therefore, $\chi_{ir}(C_n) = 5$ for $25 \leq n \leq 47$ and n is odd. ■

We have seen for $3 \leq n \leq 9$ that $\chi_{ir}(C_n) = 4$ if n is even and $\chi_{ir}(C_n) = 3$ if n is odd. If $n \geq 10$, then $\chi_{ir}(C_n) \geq 4$. We have the following conjecture.

Conjecture 3.7 *Let $k \geq 4$. If $(k-1)\binom{k-1}{2} + 1 \leq n \leq k\binom{k}{2}$, then*

$$\chi_{ir}(C_n) = \begin{cases} k & \text{if } n \neq k\binom{k}{2} - 1 \\ k+1 & \text{if } n = k\binom{k}{2} - 1. \end{cases}$$

By Propositions 3.5 and 3.6, Conjecture 3.7 is true for $10 \leq n \leq 50$. Furthermore, by Corollary 3.2, if k and n are integers satisfying the conditions in Conjecture 3.7, then $\chi_{ir}(C_n) \geq k$.

4 Graphs with Prescribed Order and Irregular Chromatic Number

We have seen that if G is a nontrivial connected graph of order n, then $2 \le \chi_{ir}(G) \le n$. Of course, a nontrivial connected graph G has chromatic number 2 if and only if G is a bipartite graph. We now present a characterization of connected graphs having irregular chromatic number 2. In order to do this, we first establish a lemma.

Lemma 4.1 *Let G be a nontrivial connected graph. Then $\chi_{ir}(G) = 2$ if and only if G is bipartite and no two vertices in the same partite set have the same degree.*

Proof. First, assume that $\chi_{ir}(G) = 2$. Then $\chi(G) = 2$ by (1) and so G is bipartite. Let c be any irregular 2-coloring of G. It remains to show that no two vertices in the same partite set have the same degree. Assume, to the contrary, that G contains two vertices u and v such that u and v belong to the same partite set of G and $\deg u = \deg v = k$ for some positive integer k. Necessarily $c(u) = c(v)$, say $c(u) = c(v) = 1$. Thus $\text{code}(u) = (1, 0, k) = \text{code}(v)$ and so c is not irregular. This is a contradiction.

For the converse, assume that G is bipartite and no two vertices in the same partite set have the same degree. Define a coloring of G by assigning color 1 to every vertex in one partite set of G and assigning color 2 to every vertex in the other partite set. By Observation 2.2, this coloring is an irregular 2-coloring of G and so $\chi_{ir}(G) = 2$. ■

For each even integer $n = 2k$, where k is a positive integer, let F_n be the bipartite graph with partite sets $X = \{x_1, x_2, \cdots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ such that $\deg x_i = \deg y_i = i$ for $1 \le i \le k$. Observe then that x_k is adjacent to all vertices in Y and x_{k-1} is adjacent to all vertices in $Y - \{y_1\}$. In general, for each integer j with $1 \le j \le k$,

$$N(x_j) = \{y_i : \quad k - j + 1 \le i \le k\}.$$

By Lemma 4.1, $\chi_{ir}(F_n) = 2$ for every positive even integer n. Figure 8 shows the graph F_8 of order 8 together with an irregular 2-coloring of F_8.

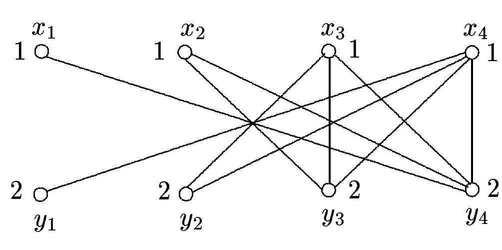

Figure 8: The graph F_8

We are now prepared to present a characterization of connected graphs with irregular chromatic number 2.

Theorem 4.2 *Let G be a connected graph of order $n \geq 2$. Then $\chi_{ir}(G) = 2$ if and only if n is even and $G \cong F_n$.*

Proof. We have seen that if $G \cong F_n$, then $\chi_{ir}(G) = 2$ by Lemma 4.1. For the converse, let G be a connected graph of order n with $\chi_{ir}(G) = 2$. It then follows by Lemma 4.1 that G is a bipartite graph such that no two vertices in the same partite set have the same degree. Let U and V be the partite sets of G. We first show that $|U| = |V|$, which implies that n is even. Assume, to the contrary, that $|U| \neq |V|$, say $|U| < |V|$. Since G is connected and all vertices of V have distinct degrees in G, it follows that $\deg v \geq 1$ for all $v \in V$ and there is at least one vertex $v' \in V$ such that $\deg v' \geq |V|$. On the other hand, since $N(v) \subseteq U$ and $|U| < |V|$, it follows that $\deg v' \leq |U| < |V|$, which is a contradiction. Therefore, $|U| = |V|$ and so n is even.

Next, we show that $G \cong F_n$. Since $|U| = |V|$ and G is connected, it follows that $1 \leq \deg x \leq k$ for all $x \in V(G)$. Furthermore,

$$\{\deg u : \ u \in U\} = \{\deg v : \ v \in V\} = \{1, 2, \ldots, k\}.$$

Assume, without loss of generality, that $U = \{u_1, u_2, \ldots, u_k\}$ and $V = \{v_1, v_2, \ldots, v_k\}$, where $\deg u_i = \deg v_i = i$ for $1 \leq i \leq k$. Therefore, u_k is adjacent to all vertices in V. Also, u_{k-1} is adjacent to all vertices in $V - \{v_1\}$ and so on. In general, for each j with $1 \leq j \leq k$, the vertex u_j is adjacent each vertex v_i for $k - j + 1 \leq i \leq k$. Therefore, $G \cong F_n$. ■

It is known that the complete graph K_n of order n is the only connected graph of order n with chromatic number n. This is not the case for irregular chromatic number.

Proposition 4.3 *If G is a complete multipartite graph of order n, then $\chi_{ir}(G) = n$.*

Proof. Let c is an irregular coloring of G and let x and y be two distinct vertices of G. If x and y belong to the same partite set of G, then $N(x) = N(y)$ and so $c(x) \neq c(y)$ by Observation 2.3. If x and y belong to the different partite sets of G, then x and y are adjacent and so $c(x) \neq c(y)$. Therefore, c must use n distinct colors and so $\chi_{ir}(G) = n$. ■

We just observed that if G is a complete multipartite graph of order n, then $N(u) = N(v)$ for every pair u, v of nonadjacent vertices of G and $\chi_{ir}(G) = n$ by Proposition 4.3. In fact, this result can be generalized to produce a characterization of connected graphs of order n with irregular chromatic number n. In order to do this, we first present a lemma.

Lemma 4.4 *Let G be a connected graph of order $n \geq 2$. Then $\chi_{ir}(G) = n$ if and only if $N(u) = N(v)$ for every pair u, v of nonadjacent vertices of G.*

Proof. First, suppose that G is a connected graph of order $n \geq 2$ such that $N(u) = N(v)$ for every pair u, v of nonadjacent vertices of G. We show that $\chi_{ir}(G) = n$. Assume, to the contrary, that $\chi_{ir}(G) \leq n-1$. Then there exists an irregular coloring c using $n-1$ or fewer colors and so there exist two vertices u and v of G such that $c(u) = c(v)$. Since c is a coloring of G, it follows that u and v are nonadjacent vertices of G. By Observation 2.3, $N(u) \neq N(v)$, which is a contradiction.

For the converse, suppose that $\chi_{ir}(G) = n$, and assume, to the contrary, that there exist two nonadjacent vertices u and v of G such that $N(u) \neq N(v)$. Define a coloring c' of G by assigning color 1 to u and v and assigning the $n-2$ colors in $\{2, 3, \cdots, n-1\}$ to the remaining $n-2$ vertices of G. Since $N(u) \neq N(v)$, it follows that code$(u) \neq$ *code*(v) and so c' is an irregular $(n-1)$-coloring of G. Thus $\chi_{ir}(G) \leq n-1$. This is a contradiction. ■

We now present a characterization of connected graphs of order n with irregular chromatic number n, which is a consequence of Proposition 4.3 and Lemma 4.4 .

Theorem 4.5 *Let G be a connected graph of order $n \geq 2$. Then $\chi_{ir}(G) = n$ if and only if G is a complete multipartite graph.*

Proof. We have seen in Proposition 4.3 that every complete multipartite graph of order n has irregular chromatic number n. Thus it remains to only verify the converse. Let G be a connected graph of order $n \geq 2$ with $\chi_{ir}(G) = n$. By Lemma 4.4 $N(u) = N(v)$ for every pair u, v of nonadjacent vertices of G. Suppose that $\chi(G) = k$, where $2 \leq k \leq n$. Let c be a k-coloring of G. Then the vertex set of G is partitioned into k color classes $V_1, V_2, \cdots, V_k$. We show that G is a complete multipartite graph with partite sets V_i for $1 \leq i \leq k$. Assume, to the contrary, that this is not the case. Then there exist two sets V_i and V_j, $1 \leq i < j \leq k$, such that some vertex u in V_i is not adjacent to some vertex v in V_j. There must be adjacent vertices $x \in V_i$ and $y \in V_j$, for otherwise all the vertices in $V_i \cup V_j$ can be colored the same, contradicting our assumption that $\chi(G) = k$. Suppose that either $x = u$ or $y = v$, say the former. Then $y \in N(u)$ and $y \notin N(v)$ and so $N(u) \neq N(v)$, which is a contradiction since u and v are not adjacent. Thus $x \neq u$ and $y \neq v$. If x is adjacent to v, then $N(u) \neq N(x)$, which is impossible since $u, x \in V_i$. Thus v and x are not adjacent. However then, $N(v) \neq N(x)$, which again is impossible. ■

By Theorem 4.2, there is no connected graph of odd order having irregular chromatic number 2. On the other hand, every pair k, n of integers with $3 \leq k \leq n$ is realizable as the irregular chromatic number and the order of some connected graph, as we show next.

Theorem 4.6 *For every pair k, n of integers with $3 \leq k \leq n$, there exists a connected graph of order n having irregular chromatic number k.*

Proof. We consider two cases, according to whether $k \geq \sqrt{n}$ or $k < \sqrt{n}$.

Case 1. $k \geq \sqrt{n}$. Then $n-k \leq k^2-k$. We construct a graph G from the complete graph K_k by adding $n-k$ new vertices to K_k in such a way that every new vertex is adjacent to exactly one vertex of K_k and each vertex of K_k is adjacent to at most $k-1$ new vertices. Thus the order of G is n. We show that $\chi_{ir}(G) = k$. Since no two vertices in $V(K_k)$ can be colored the same, $\chi_{ir}(G) \geq k$. To show that $\chi_{ir}(G) \leq k$, we define an irregular k-coloring of G. Let $V(K_k) = \{v_1, v_2, \ldots, v_k\}$ and let $U = V(G) - V(K_k)$ be the set of new vertices of G. For each i with $1 \leq i \leq k$, let $U_i = N(v_i) \cap U$. Thus $|U_i| \leq k-1$ for all i with $1 \leq i \leq k$. We now assign color i to the vertex v_i ($1 \leq i \leq k$) and assign $|U_i|$ distinct colors from the set $\{1, 2, \ldots, k\} - \{i\}$ to the vertices in U_i for $1 \leq i \leq k$. Observe that if $u, v \in V(G)$ such that $c(u) = c(v)$, then either (1) u and v are end-vertices of G and u and v are adjacent to two distinct vertices in $V(K_k)$ or (2) $u \in V(K_k)$ and v is an end-vertex of G. In (1), u and v are adjacent to vertices having a different color; while in (2), u and v have different degree. In either case, $\text{code}(u) \neq \text{code}(v)$. Since distinct vertices have distinct color codes, this coloring is an irregular k-coloring of G and so $\chi_{ir}(G) \leq k$. Therefore, $\chi_{ir}(G) = k$.

Case 2. $k < \sqrt{n}$. We consider two subcases, according to whether k and n are of the same parity or of opposite parity.

Subcase 2.1. *k and n are of the same parity.* Then $n-k+2$ is even and so $n-k+2 = 2a$ for some positive integer a. Let F be the bipartite graph of order $n-k+2$ with irregular chromatic number 2 described in Theorem 4.2. Suppose that $X = \{x_1, x_2, \cdots, x_a\}$ and $Y = \{y_1, y_2, \cdots, y_a\}$ are the partite sets of F with $\deg_F x_i = \deg_F y_i = i$ for $1 \leq i \leq a$. Define the graph G by adding $k-2$ new vertices $z_1, z_2, \ldots, z_{k-2}$ to F and joining each vertex z_i ($1 \leq i \leq k-2$) to the vertex x_a of F. Then the order of G is n. We show that $\chi_{ir}(G) = k$. Since x_a is adjacent to $k-1$ end-vertices, namely $y_1, z_1, z_2, \ldots, z_{k-2}$, it follows that $\chi_{ir}(G) \geq k$. Next, we define a k-coloring c of G by $c(x_i) = 2$ and $c(y_i) = 1$ for $1 \leq i \leq a$, and $c(z_j) = j+2$ for $1 \leq j \leq k-2$. Observe that if $u, v \in V(G)$ such that $c(u) = c(v)$, then u and v belong to the same partite set of F and so $\deg_G u \neq \deg_G v$, implying that $\text{code}(u) \neq \text{code}(v)$. Since distinct vertices have distinct color codes, c is an irregular k-coloring of G and so $\chi_{ir}(G) \leq k$. Therefore, $\chi_{ir}(G) = k$.

Subcase 2.2. *k and n are of opposite parity.* Then $n-k+1$ is even and so $n-k+1 = 2b$ for some positive integer b. Let H be the bipartite graph of order $n-k+1$ with irregular chromatic number 2 described in Theorem 4.2. Suppose that $X = \{x_1, x_2, \cdots, x_b\}$ and $Y = \{y_1, y_2, \cdots, y_b\}$ are the partite sets of H with $\deg_H x_i = \deg_H y_i = i$ for $1 \leq i \leq b$. Define the graph G by adding $k-1$ new vertices $z_1, z_2, \ldots, z_{k-1}$ to H and joining each vertex z_i ($1 \leq i \leq k-1$) to the vertex x_1. Then the order of G is n. We show that $\chi_{ir}(G) = k$. Since x_1 is adjacent to $k-1$ end-vertices, it follows that $\chi_{ir}(G) \geq k$. Next we define a k-coloring c of G by $c(x_i) = 1$ and $c(y_i) = 2$ for $2 \leq i \leq a$, $c(z_j) = j$ for $1 \leq j \leq k-1$, and $c(x_1) = c(y_1) = k$. Let $u, v \in V(G)$ such that $c(u) = c(v)$, then either (1) u and v belong to the same partite set of H, (2) $\{u, v\} = \{x_1, y_1\}$, or (3) one of u and v belongs to H and the other is an end-vertex of G. In any case, $\deg_G u \neq \deg_G v$ and so $\text{code}(u) \neq \text{code}(v)$. Since distinct vertices have distinct color codes, c is an irregular k-coloring of G and so $\chi_{ir}(G) \leq k$. Therefore, $\chi_{ir}(G) = k$. ■

Combining Theorems 4.2 and 4.6, we have the following.

Corollary 4.7 *Let k and n be integers with $2 \leq k \leq n$. Then there exists a connected graph of order n having irregular chromatic number k if and only if $(k, n) \neq (2, n)$, where n is an odd integer.*

We have seen that if G is a nontrivial connected graph with $\chi(G) = a$ and $\chi_{ir}(G) = b$, then $2 \leq a \leq b$. Next we show that every pair a, b of integers with $2 \leq a \leq b$ is realizable as the chromatic number and irregular chromatic number of some connected graph.

Proposition 4.8 *For every pair a, b of integers with $2 \leq a \leq b$, there is a connected graph G with $\chi(G) = a$ and $\chi_{ir}(G) = b$.*

Proof. If $a = b$, then for $G = K_a$ we have $\chi(G) = \chi_{ir}(G) = a$ by Proposition 4.3. If $2 = a < b$, then for $G = K_{1,b-1}$ we have $\chi(G) = 2$ and $\chi_{ir}(G) = b$ by Proposition 4.3 as well. Thus, we may assume that $3 \leq a < b$. Let G be the graph obtained from the complete graph K_a with $V(K_a) = \{u_1, u_2, \ldots, u_a\}$ by adding $b-1$ new vertices $v_1, v_2, \cdots, v_{b-1}$ to K_a and joining each vertex v_i $(1 \leq i \leq b-1)$ to the vertex u_1 of K_a. Then $\chi(G) = a$. To show that $\chi_{ir}(G) = b$, observe that if c is an irregular coloring of G, then (1) $c(u_1) \neq c(v_p)$ for $1 \leq p \leq b-1$ since $u_1 v_p \in E(G)$ and (2) $c(v_p) \neq c(v_q)$ for $1 \leq p \neq q \leq b-1$ since $N(v_p) = N(v_q)$. Thus $\chi_{ir}(G) \geq b$. On the other hand, the coloring c' of G defined by $c'(u_i) = i$ for $1 \leq i \leq a$ and $c'(v_p) = i+1$ for $1 \leq i \leq b-1$ is an irregular b-coloring of P. Thus $\chi_{ir}(G) \leq b$ and so $\chi_{ir}(G) = b$. ■

References

[1] M. Aigner and E. Triesch, Irregular assignments and two problems á la Ringel. *Topics in Combinatorics and Graph Theory.* (R. Bodendiek and R. Henn, eds.). Physica, Heidelberg (1990) 29–36.

[2] M. Aigner, E. Triesch and Z. Tuza, Irregular assignments and vertex-distinguishing edge-colorings of graphs. *Combinatorics '90: Recent Trends and Applications* (A. Barlotti, A. Bichara, P. V. Ceccherini, and G. Tallini, eds.). Elsevier Science Pub., New York (1992) 1–9.

[3] M. Albertson and K. Collins, Symmetric breaking in graphs. *Electron. J. Combin.* **3** (1996), R18.

[4] A. C. Burris, On graphs with irregular coloring number 2. *Congr. Numer.* **100** (1994) 129–140

[5] A. C. Burris, The irregular coloring number of a tree. *Discrete Math.* **141** (1995) 279–283.

[6] G. Chartrand, H. Escuadro, F. Okamoto, and P. Zhang, Detectable colorings of graphs. *Util. Math.* To appear.

[7] G. Chartrand, L. Lesniak, D. W. VanderJagt, and P. Zhang, Recognizable colorings of graphs. Preprint.

[8] G. Chartrand and P. Zhang, *Introduction to Graph Theory.* McGraw-Hill, Boston (2005).

[9] R. C. Entringer and L. D. Gassman, Line-critical point determining and point distinguishing graphs. *Discrete Math.* **10** (1974) 43-55.

[10] D. Erwin and F. Harary, Destroying automorphisms by fixing nodes. Preprint.

[11] F. Harary, Methods of destroying the symmetries of a graph. *Bull. Malays. Math. Sci. Soc.* **24** (2001) 183-191.

[12] F. Harary and R. A. Melter, On the metric dimension of a graph. *Ars Combin.* **2** (1976) 191-195.

[13] F. Harary and M. Plantholt, The point-distinguishing chromatic index. *Graphs and Applications.* Wiley, New York (1985) 147-162.

[14] P. J. Slater, Leaves of trees. *Congress. Numer.* **14** (1975) 549-559.

[15] D. P. Sumner, Point determination in graphs. *Discrete Math.* **5** (1973) 179-187.