THE SPECTRA OF MULTIPLICATIVE ATTRIBUTE GRAPHS

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Abstract. A multiplicative attribute graph is a random graph in which vertices are represented by random words of length t in a finite alphabet Γ, and the probability of adjacency is a symmetric function $\Gamma^t \times \Gamma^t \to [0,1]$. These graphs are a generalization of stochastic Kronecker graphs, and both classes have been shown to exhibit several useful real world properties. We establish asymptotic bounds on the spectra of the adjacency matrix and the normalized Laplacian matrix for these two families of graphs under certain mild conditions. As an application we examine various properties of the stochastic Kronecker graph and the multiplicative attribute graph, including the diameter, clustering coefficient, chromatic number, and bounds on low-congestion routing.

1. Introduction

Over the last few decades, spurred by the attempts to understand and model modern complex networks, there has been an extensive amount of literature devoted to studying models for random graphs that differ significantly from the standard Erdős-Rényi random graph and the d-regular random graph (see for instance [5, 9, 11, 18, 30]). One trend that has begun to emerge among these random graph models is the use of mathematical primitives to create models for complex networks that exhibit complicated behavior while still being analytically tractable. For example, inhomogeneous random graphs [3, 4] and random dot product graphs [26, 32, 41, 44, 45], both build graphs over an inner product space and use the inner product to govern the edge connectivity, while stochastic Kronecker graphs [28, 29] and multiplicative attribute graphs [25] use the Kronecker product of matrices to control the edge probabilities. In this work we focus on the spectral properties of the latter two random graph models, showing that these properties can be derived in a natural way using the Kronecker product.

More formally, the Kronecker product of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is a matrix $A \otimes B = C \in \mathbb{R}^{mp\times nq}$ where $C_{i,j} = A_{\lceil \frac{i}{m} \rceil, \lceil \frac{j}{n} \rceil} B_{i \bmod p, j \bmod q}$ and $x \bmod p \in [p] = \{1, 2, \ldots, p\}$. That is,

$$A \otimes B = C = \begin{bmatrix} A_{1,1}B & A_{1,2}B & \cdots & A_{1,n}B \\ A_{2,1}B & A_{2,2}B & \cdots & A_{2,n}B \\ \dots & \dots & \ddots & \dots \\ A_{m,1}B & A_{m,2}B & \cdots & A_{m,n}B \end{bmatrix}.$$

A stochastic Kronecker graph is formed by taking a symmetric $k\times k$ matrix P with entries in the interval $[0,1]$ and a positive integer t, and forming the t-fold Kronecker product, denoted $P^{\otimes t}$. Each edge $\{i,j\}$ is then present independently with probability $P^{\otimes t}_{i,j} = P^{\otimes t}_{j,i}$. We will say that such a graph is a t^{th}-order stochastic Kronecker graph with generating matrix P. Recently, the stochastic Kronecker graph as been advanced as a model for the internet and other complex networks [29] especially in the case where the generating matrix is $\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ and $\alpha \geq \beta \geq \gamma$. Mahdian and Xu have recently analyzed the connectivity, diameter, and emergence of the giant component in this context [31] while the first author and Horn analyzed the emergence of the giant component of a general 2×2 generating matrix [37]. The multiplicative attribute graph is a natural generalization of stochastic Kronecker graphs to allow multiple copies of each vertex before determining the random edges. In order to make this precise, we equip the $k \times k$ generating matrix for the stochastic Kronecker graph with an alphabet Γ of size k, and define a function $w : V \to \Gamma^t$. Then any two vertices $u, v \in V$ are connected independently with probability $P^{\otimes t}_{w(u),w(v)}$. We note here that u may be equal to v,

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so that we allow self-edges with the appropriate probability. If the function w is a bijection, then this is a t^{th}-order stochastic Kronecker graph with generating matrix P, while if w is not a bijection we say that the resulting graph is a t^{th}-order multiplicative attribute graph with generating matrix P.

We provide here asymptotic bounds on the spectra of both the stochastic Kronecker graph and a generalization due to Kim and Leskovec [25] known as the multiplicative attribute graph. Moreover, we show some applications of these bounds to graph properties related to spectra. Although there are several natural spectra of graphs to consider, we focus on two in particular, the spectrum of the adjacency matrix A and the normalized Laplacian $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$, where D is the diagonal matrix of degrees. For a given graph G we will denote the adjacency matrix by $A(G)$ and the normalized Laplacian by $\mathcal{L}(G)$. These spectra are of particular interest in the analysis of complex and real world networks as they provide insight to fundamental structural properties of the network.

2. Preliminaries

We begin with an overview of the structural properties related to graph spectra and our main tools for the analysis of the spectra for stochastic Kronecker graphs and multiplicative attribute graphs. Given a graph G we consider first the structural properties of G determined by the spectrum of its adjacency matrix A. To that end let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the spectrum (with multiplicities) of the adjacency matrix A.

It is well known that A^t_{ij} represents the number of walks of length t from i to j in the graph G. Furthermore, this implies that $\operatorname{trace}(A^t)$ is the total number of walks of length t in the graph, and thus the number of edges is in G is $\frac{1}{2}\operatorname{trace}(A^2) = \frac{1}{2}\sum_i \mu_i^2$, the total number of triangles in G is $\frac{1}{6}\operatorname{trace}(A^3) = \frac{1}{6}\sum_i \mu_i^3$, and the number of 4 cycles in G is

$$\frac{1}{8}\left(\operatorname{trace}(A^4) - 2\operatorname{trace}(A^2)^2 + \operatorname{trace}(A^2)\right) = \frac{1}{8}\left(\sum_i \mu_i^4 - 2\left(\sum_j \mu_j^2\right)^2 + \sum_k \mu_k^2\right).$$

The first of these two allow us to calculate a measure of the clustering of the graph G based purely on the spectrum of the adjacency matrix. More specifically, in the study of complex networks an important parameter of the network is the clustering coefficient C of the graph (for an overview, see for instance [6, 10, 24]) which may be variously defined, but is an attempt to capture how often short paths are closed into cycles. If T is the number of triangles in the graph, then one definition of the clustering coefficient is $\frac{3T}{\sum_v \binom{\deg(v)}{2}}$, that is, the fraction of adjacent pairs of edges that are part of a triangle. It is easy to see that under this definition of the clustering coefficient we have

$$C = \frac{\operatorname{trace}(A^3)}{\operatorname{trace}(A^2)^2 - \operatorname{trace}(A^2)}.$$

The spectrum of the adjacency matrix also famously gives information about the chromatic number via result of Wilf [43] and Hoffman [21, 22], which combine to give that

$$1 - \frac{\mu_n}{\mu_1} \leq \chi(G) \leq 1 + \mu_n.$$

Although not directly applicable to the case of stochastic Kronecker graphs and multiplicative attribute graphs the spectrum of the adjacency matrix of G and its compliment also give some control over the presence of a small cycles [33] and the Hamiltonicity of G [17]. For a reference to other properties of the adjacency matrix we refer the reader to the manuscript of Brouwer and Haemers [7].

We now consider the properties of the spectrum of the normalized Laplacian $\mathcal{L}$, which has eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$. In many ways the consequences of the spectrum are either directly or indirectly tied to the ability of the spectral gap $\lambda = \max_{i\neq 1}|1-\lambda_i| = \max\{|1-\lambda_2|, |1-\lambda_n|\}$ to control the mixing rate of the uniform random walk on the underlying graph G. Specifically, let π be the limiting distribution of the uniform random walk on G (if it exists[1]), then the relative point-wise distance of the s^{th} power of the

[1]It is a standard exercise that this distribution exists and is unique if the underlying undirected graph G is connected and is not bipartite, see for instance [15].

transition matrix P to the limiting distribution is

$$\Delta(s) = \max_{x,y} \frac{|P^s(y,x) - \pi(x)|}{\pi(x)}.$$

Now, for any subset X of the vertices we denote by $\mathrm{vol}\,(X) = \sum_{x\in X} \deg(x)$, then we have (as given in [15])

$$\Delta(s) = e^{-s(1-\lambda)} \frac{\mathrm{vol}\,(G)}{\min_v \deg(v)}$$

and as a consequence after at most $\frac{1}{1-\lambda}\ln\left(\frac{\mathrm{vol}(G)}{\epsilon \min_v \deg v}\right)$ steps of the uniform random walk the relative point-wise distance is at most ϵ. As we will see in a Theorem 3 and Theorem 4, if the parameters of the stochastic Kronecker and multiplicative attribute graph are such that with high probability the graph is connected, then the spectral gap is asymptotically constant, yielding rapid mixing of the uniform random walk. Although this application of the spectrum of the normalized Laplacian has significant applications in the broader context of algorithms on graphs, we will focus instead on specific structural properties that are consequences of the spectrum of the normalized Laplacian and more relevant to stochastic Kronecker graphs and multiplicative attribute graphs in reference to their potential application as models for complex networks.

One of the identifying properties of complex networks is a relatively short diameter compared to the size of the network, or more prosaically the "six degrees of Kevin Bacon" property [19]. By work of Chung [14] we have that diameter of a graph can be bounded above by $\left\lceil \frac{\ln(n-1)}{\ln\left(\frac{\lambda_n+\lambda_2}{\lambda_n-\lambda_2}\right)} \right\rceil$, which in the case of constant spectral gap gives asymptotically logarithmic diameter. As we will see in Theorem 6 and Theorem 7, for some settings of the generating matrix, this result is sufficient to give a constant upper bound on the diameter of a stochastic Kronecker or multiplicative attribute graph.

A significant amount of interest in complex networks has been driven by the ability of networks (such as the physical layer of the internet) to handle ever increasing demand well beyond earlier projected capacity. Much of this ability can be explained by the following result which appears in [15] and is adapted from the work of Alon, Chung, and Graham in [2]:

Theorem 1. *Let G be a graph on n vertices. Let $A = \{(x_i, y_i) : x_i \in X, y_i \in Y\}$ denote any assignment such that each vertex v is in X with multiplicity $\deg v$ and in Y with multiplicity $\deg v$. There are paths P_i joining x_i to y_i of length at most $\frac{2}{\lambda_i}\ln(n)$ such that each edge of G is contained in at most $\frac{20}{\lambda_1}\ln(n)$ paths P_i.*

Finally, we would be remiss if we failed to mention the connection between the spectrum of the normalized Laplacian and the isoperimetric properties of the graph G. The most famous of these connections is to the Cheeger constant or the conductance, defined as

$$\Phi = \min_{S\subset V(G)} \frac{e(S,\overline{S})}{\min\left\{\mathrm{vol}\,(S), \mathrm{vol}\,\left(\overline{S}\right)\right\}},$$

where $e(X,Y)$ is the number of edges with one end point in X and the other in Y. Via results of Jerrum and Sinclar [39] and Sokal [27], we know that $2\Phi \geq \lambda_2 \geq \frac{\Phi^2}{2}$ and thus the spectrum gives reasonable control on the number of edges crossing cuts in the graph. This control can be refined to the discrepancy property [15] which gives that for any two subsets of vertices X and Y,

$$\left| e(X,Y) - \frac{\mathrm{vol}\,(X)\,\mathrm{vol}\,(Y)}{\mathrm{vol}\,(G)} \right| \leq \lambda \frac{\sqrt{\mathrm{vol}\,(X)\,\mathrm{vol}\,(Y)\,\mathrm{vol}\,\left(\overline{X}\right)\mathrm{vol}\,\left(\overline{Y}\right)}}{\mathrm{vol}\,(G)}.$$

With these applications in mind, let us turn now to the tools we shall use to analyze the spectra of the stochastic Kronecker graph and multiplicative attribute graph. Our primary tool will be the following result of the first author and Chung.

Theorem 2 ([36])**.** *Let G be a random graph with independent edges generated according to the matrix $\mathcal{P}$ and let A be the associated adjacency matrix. Let D be the diagonal matrix of expected degrees and let Δ and δ denote the maximum and minimum expected degrees, respectively. If $\Delta > \frac{4}{9}\ln\left(\frac{2n}{\epsilon}\right)$ and n is sufficiently* *large, then with probability at least* $1-\epsilon$ *we have that for all* i

$$|\lambda_i(A) - \lambda_i(\mathcal{P})| \leq \sqrt{4\Delta \ln\left(\frac{2n}{\epsilon}\right)}.$$

Additionally, if $\delta \geq 3\ln\left(\frac{4n}{\epsilon}\right)$*, then with probability at least* $1-\epsilon$ *we have that, for all* i

$$\left|\lambda_i\left(\mathcal{L}(G)\right) - \lambda_i\left(I - D^{-1/2}\mathcal{P}D^{-1/2}\right)\right| \leq 3\sqrt{\frac{3\ln\left(\frac{4n}{\epsilon}\right)}{\delta}}.$$

We note that this theorem is derived from a matrix Chernoff inequality. Previously, various matrix concentration inequalities have been derived by many authors including Ahlswede-Winter [1], Cristofides-Markström [8], Oliveira [34], Gross [20], Recht [38], and Tropp [40]. Oliveira [34] also gives a slightly weaker version of Theorem 2.

As we can see from Theorem 2, in both the case of the adjacency matrix and the normalized Laplacian, the spectrum of G is heavily influenced by the underlying probability matrix $\mathcal{P}$, in our cases $P^{\otimes t}$. Thus we recall the following properties of the Kronecker power of a matrix.

Observation 1. If M has eigenvalues $\lambda_1, \ldots, \lambda_k$, then for every non-negative integer solution to $a_1 + \cdots + a_k = t$, $M^{\otimes t}$ has the eigenvalue $\lambda_1^{a_1} \cdots \lambda_k^{a_k}$ with multiplicity $\binom{t}{a_1, a_2, \ldots, a_k}$.

Observation 2. For matrices A and B and any integer t, we have $(AB)^{\otimes t} = A^{\otimes t}B^{\otimes t}$.

For other standard properties of the Kronecker product see [42].

3. Stochastic Kronecker Graphs

Although the application of Theorem 2 to stochastic Kronecker graphs can viewed as an easy consequence of the properties of Kronecker multiplication, we provide the full details here as a prelude to the analysis for multiplicative attribute graphs.

Theorem 3. *Let* G *be a* t^{th}*-order stochastic Kronecker graph over the alphabet* Γ *of size* k *with affinity matrix* P*, and let* $n = k^t$*. Let* D *be the diagonal matrix of column sums of* P *and let* δ *and* Δ *be the minimum and maximum diagonal entries of* D*, respectively. Let* $\epsilon > 0$*. If* $\Delta > 1$ *and* $t \geq \frac{\ln\left(\frac{4}{9}\ln\left(\frac{2n}{\epsilon}\right)\right)}{\ln(\Delta)}$*, then for all* $i \in [n]$*,*

$$\left|\lambda_i(A(G)) - \lambda_i\left(P^{\otimes t}\right)\right| \leq \sqrt{4\Delta^t \ln\left(\frac{2n}{\epsilon}\right)},$$

with probability at least $1-\epsilon$*.*

If $\delta > 1$ *and* $t \geq \frac{\ln\left(3\ln\left(\frac{4n}{\epsilon}\right)\right)}{\ln(\delta)}$*, then for all* $i \in [n]$*,*

$$\left|\lambda_i(\mathcal{L}(G)) - \left(1 - \lambda_{n+1-i}\left(\left(D^{-1/2}PD^{-1/2}\right)^{\otimes t}\right)\right)\right| \leq 3\sqrt{\frac{3\ln\left(\frac{4n}{\epsilon}\right)}{\delta^t}},$$

with probability at least $1-\epsilon$*.*

Before proving Theorem 3 we comment briefly on the lower bounds required on t. Consider first the bound on t for the spectrum of the adjacency matrix, as $n = k^t$ this condition may be rewritten as $\Delta^t \geq \frac{4}{9}\ln\left(\frac{2k^t}{\epsilon}\right)$. Solving for t when $\Delta^t = \frac{4}{9}\ln\left(\frac{2k^t}{\epsilon}\right)$ we have

$$t = \log_k\left(\frac{\epsilon}{2}\right) - \frac{1}{\ln(\Delta)}\mathcal{W}\left(-\frac{9}{4}\log_k(\Delta)\Delta^{-\log_k\left(\frac{\epsilon}{2}\right)}\right)$$

where $\mathcal{W}$ is the Lambert-W function and $\mathcal{W}(z)e^{\mathcal{W}(z)} = z$ (see [16, Section 4.13] for a reference of properties of the Lambert-W function). Now if $z \to 0^-$ then $\mathcal{W}(z) = -\eta - \ln(\eta) + \mathcal{O}\left(\frac{\ln(\eta)}{\eta}\right)$ where $\eta = -\ln(-z)$. In our case $\eta = -\log_k\left(\frac{\epsilon}{2}\right)\ln(\Delta) + \ln\left(\frac{4}{9\log_k(\Delta)}\right)$. Thus we have

$$\begin{aligned}
t &= \log_k\left(\frac{\epsilon}{2}\right) - \frac{1}{\ln(\Delta)}\left(\log_k\left(\frac{\epsilon}{2}\right)\ln(\Delta) - \ln\left(\frac{4}{9\log_k(\Delta)}\right) - \ln\ln\left(\frac{4}{9\log_k(\Delta)}\Delta^{-\log_k\left(\frac{\epsilon}{2}\right)}\right) + o(1)\right) \\
&= \frac{1}{\ln(\Delta)}\ln\left(\frac{4}{9\log_k(\Delta)}\right) + \frac{1}{\ln(\Delta)}\ln\ln\left(\frac{4}{9\log_k(\Delta)}\Delta^{-\log_k\left(\frac{\epsilon}{2}\right)}\right) + o(1) \\
&= \log_\Delta\left(\frac{4}{9\log_k(\Delta)}\right) + \frac{1}{\ln(\Delta)}\ln\left(\ln\left(\frac{4}{9\log_k(\Delta)}\right) - \ln(\Delta)\log_k\left(\frac{\epsilon}{2}\right)\right) + o(1) \\
&= \log_\Delta\left(\frac{4}{9\log_k(\Delta)}\right) + \frac{1}{\ln(\Delta)}\ln\left(\log_k\left(\frac{2}{\epsilon}\right)(\ln(\Delta) - o(1))\right) + o(1) \\
&= \log_\Delta\left(\frac{4}{9\log_k(\Delta)}\right) + \frac{1}{\ln(\Delta)}\ln\left(\log_k\left(\frac{2}{\epsilon}\right)\right) + \frac{1}{\ln(\Delta)}\ln\ln\left(\Delta - o(1)\right) + o(1) \\
&= \log_\Delta\left(\log_k\left(\frac{2}{\epsilon}\right)\right) + \log_\Delta\left(\frac{4\ln(\Delta)}{9\log_k(\Delta)}\right) + o(1) \\
&= \log_\Delta\left(\log_k\left(\frac{2}{\epsilon}\right)\right) + \log_\Delta\left(\frac{4\ln(k)}{9}\right) + o(1) \\
&= \log_\Delta\left(\frac{4\ln\left(\frac{2}{\epsilon}\right)}{9}\right) + o(1)\,.
\end{aligned}$$

Thus for ϵ sufficiently small it suffices for $t > \log_\Delta\left(\frac{4}{9}\ln\left(\frac{2}{\epsilon}\right)\right)$. A similar statement holds for the condition on t for the normalized Laplacian, namely for ϵ sufficiently small it suffices for $t > \log_\delta\left(3\ln\left(\frac{4}{\epsilon}\right)\right)$. However, in the case of the normalized Laplacian the bound on t is essentially superfluous as the entire spectrum is contained in $[0,2]$ and if $t < \frac{\ln\left(3\ln\left(\frac{2n}{\epsilon}\right)\right)}{\ln(\delta)}$, then the error bound given is at least $\sqrt{27} > 2$.

Proof. We first consider the degree of an arbitrary vertex v with $w(v) = \sigma = \sigma_1\sigma_2\cdots\sigma_t$. For any $s \geq 1$ and each $\gamma \in \Gamma^s$ let d_γ be the appropriate column sum of $P^{\otimes s}$. Using this notation we have

$$\mathbb{E}\left[\deg(v)\right] = \sum_{\tau\in\Gamma^t} P^{\otimes t}_{\omega,\tau} = \sum_{\tau\in\Gamma^t}\prod_{i=1}^{t} P_{\sigma_i,\tau_i} = \prod_{i=1}^{t}\sum_{\gamma\in\Gamma} P_{\sigma_i,\gamma} = \prod_{i=1}^{t} d_{\sigma_i}.$$

Thus the expected minimum degree is δ^t and the expected maximum degree is Δ^t. Thus, if $\delta > 1$ or $\Delta > 1$ we can apply the appropriate results from Theorem 2. In order to finish the result it suffices to note by Observation 2 that $(D^{-1/2}PD^{-1/2})^{\otimes t} = (D^{-1/2})^{\otimes t}P^{\otimes t}(D^{-1/2})^{\otimes t}$, and thus for any $\sigma,\tau \in \Gamma^t$ with expected degrees d_σ and d_τ, we have $\left(D^{-1/2}PD^{-1/2}\right)^{\otimes t}_{\sigma,\tau} = \frac{P^{\otimes t}_{\sigma,\tau}}{\sqrt{d_\sigma d_\tau}}$. □

4. Multiplicative Attribute Graphs

Although the stochastic Kronecker graph model has been proposed as a model for a variety of different complex networks, it has the significant drawback that with a $k \times k$ generating matrix, it is only possible to generate a network whose size is a power of k. There are several natural methods to attempt to bootstrap a stochastic Kronecker graph to a network of arbitrary size, but it is not clear *a priori* that these methods preserve the theoretical properties of stochastic Kronecker graphs. One such method, proposed by Kim and Leskovec, is the multiplicative attribute graph [25]. In this model there is an alphabet Γ, a generating matrix P, an order t, and a function $w\colon V \to \Gamma^t$, which associates to each vertex in V a word in Γ^t. We define the signature function $n\colon \Gamma^t \to \mathbb{N}$ by $n(\sigma) = \left|w^{-1}(\sigma)\right|$. That is, $n(\sigma)$ indicates how many vertices are associated with the word σ. For simplicity of notation, we will write $n_\sigma = n(\sigma)$. Note that the resulting graph has $n = \sum_{\sigma\in|\Gamma|^t} n_\sigma$ vertices. Given these functions, each edge is present independently with the probability of u and v being adjacent equal to $P^{\otimes t}_{w(u),w(v)}$. As with the stochastic Kronecker graph, we allow $u = v$, so that this graph may have loops. We will call the collection $\{n_\sigma\}_{\sigma\in\Gamma^t}$ the signature of the graph G.

We will restrict ourselves to the case where the signature of the multiplicative attribute graph is determined randomly and by a probability distribution over Γ. Specifically, if Q is a diagonal matrix representing the probability distribution over Γ, in the t^{th}-order multiplicative attribute graph, each vertex is assigned the word $\sigma \in \Gamma^t$ independently with probability $Q^{\otimes t}_{\sigma,\sigma}$. Without loss of generality, we may assume that for all $\gamma \in \Gamma$, $Q_{\gamma,\gamma} > 0$ as otherwise we may consider the smaller alphabet $\Gamma' = \Gamma - \{\gamma\}$. In the following theorem we show that the qualitative behavior of the spectral properties of multiplicative attribute graph are essentially that of an appropriate stochastic Kronecker graph plus some nearly trivial eigenvalues. In particular, if the distribution on Γ is uniform then (up to trivial eigenvalues) the spectrum of a t^{th}-order multiplicative attribute graph is the essentially the same as that of the t^{th}-order stochastic Kronecker graph with the same generating matrix.

Theorem 4. *Let G be a t^{th}-order multiplicative attribute graph on n vertices over the alphabet Γ of size k, with probability matrix Q and affinity matrix P. Let D be the diagonal matrix of column sums of QP and let δ be the minimum diagonal entry of D and $q_{\min}$ be the minimum diagonal entry of Q. Let $\epsilon > 0$. If*

$$t \leq \min\left\{ \frac{\ln(n) - \ln\left(6\ln\left(\frac{8n}{\epsilon}\right)\right)}{\ln\left(\frac{1}{\delta}\right)}, \frac{\ln(n) - \ln\left(12\ln\left(\frac{2n}{\epsilon}\right)\right)}{\ln\left(\frac{1}{q_{\min}}\right)} \right\}, \tag{1}$$

then with probability at least $1-\epsilon$ there is a set $A \subset [n]$, where $A = \{a_1, \ldots, a_{k^t}\}$, such that for $i \in [k^t]$,

$$\begin{aligned}\left|\lambda_{a_i}(\mathcal{L}(G)) - \left(1 - \lambda_{k^t+1-i}\left(\left(D^{-1/2}Q^{1/2}PQ^{1/2}D^{-1/2}\right)^{\otimes t}\right)\right)\right| &\leq 3\sqrt{\frac{6\ln\left(\frac{8n}{\epsilon}\right)}{\delta^t n}} + 4\sqrt{\frac{3\ln\left(\frac{2n}{\epsilon}\right)}{q_{\min}^t n}} \\ &\leq 15\sqrt{\frac{\ln\left(\frac{8n}{\epsilon}\right)}{n\min\{q_{\min}^t, \delta^t\}}}.\end{aligned}$$

Furthermore, for all $j \notin A$, $|\lambda_j(\mathcal{L}(G)) - 1| \leq 3\sqrt{\frac{6\ln\left(\frac{8n}{\epsilon}\right)}{\delta^t n}}$.

The precise conditions on t in (1) will fall out from the proof. We note that as with the bound in Theorem 3, this can be transformed into a bound on n in the case that $t \to \infty$, specifically

$$n \geq \max\left\{ \frac{6\ln\left(\frac{48}{\epsilon\delta^t}\right)}{\delta^t}, \frac{12\ln\left(\frac{24}{\epsilon q_{\min}^t}\right)}{q_{\min}^t} \right\} + o(1).$$

This should be unsurprising, as when n is sufficiently large, we will have close to the expected number of vertices for each $\sigma \in \Gamma^t$, and we can thus use concentration techniques on the expected signature. Alternatively, we note that there is some ρ, depending only on $\delta, q_{\min}$, and ϵ, such that if $t \leq \rho\ln(n)$ and n is sufficiently large the conditions for the theorem hold.

Proof. In order to analyze the spectrum of G we consider the random assignment of vertices to words in Γ^t separately from the random generation of edges.

Fix the signature $\{n_\sigma\}_{\sigma\in\Gamma^t}$ of the graph G, and define for each $\sigma \in \Gamma^t$, $d_\sigma = \sum_{\tau\in\Gamma^t} n_\tau P^{\otimes t}_{\sigma,\tau}$. We note that for any vertex v, $\mathbb{E}[\deg v] = d_{w(v)}$ and thus the minimum expected degree is $d_{\min} = \min_{\sigma\in\Gamma^t} d_\sigma$ for graphs with a fixed signature. Now using Theorem 2, if $d_{\min} \geq 3\ln\left(\frac{8n}{\epsilon}\right)$, then with probability at least $1 - \frac{\epsilon}{2}$, $|\lambda_i(\mathcal{L}(G)) - (1 - \lambda_{n-i+1}(M))| \leq 3\sqrt{\frac{3\ln\left(\frac{8n}{\epsilon}\right)}{d_{\min}}}$ for all i, where

$$M_{u,v} = \frac{P^{\otimes t}_{w(u),w(v)}}{\sqrt{d_{w(u)}d_{w(v)}}}.$$

In order to understand the spectrum of M, we consider the case where every element of the signature is at least 1, that is, $n_\sigma \geq 1$ for all $\sigma \in \Gamma^t$. Thus for every $\sigma \in \Gamma^t$, there exists a vertex $v \in V$ such that $w(v) = \sigma$. We will abuse notation and refer to any such vertex as $w^{-1}(\sigma)$. Hence we may define the $k^t \times k^t$ matrix by $\tilde{M}_{\sigma,\tau} = \sqrt{n_\sigma n_\tau} M_{w^{-1}(\sigma),w^{-1}(\tau)}$. We claim that $\tilde{M}$ captures the non-trivial portion of the spectrum of M.

Observe that for any two vertices u and v with $w(u) = w(v)$, the corresponding rows of M are identical, and thus for each $\sigma \in \Gamma^t$, M has an eigenvalue of 0 of multiplicity $n_\sigma - 1$. Hence, the multiplicity of 0 as an eigenvalue of M is at least $n - k^t$. In order to show that the remaining eigenvalues of M are the spectrum of $\tilde{M}$, let φ be an eigenvector for M with corresponding eigenvalue $\lambda \neq 0$. Define $\psi \in \mathbb{R}^{k^t}$ by

$\psi(\sigma) = \frac{1}{n_\sigma}\sum_{w(v)=\sigma}\varphi(v)$. Now for any $v \in V(G)$ with $w(v) = \sigma$, $\lambda\varphi(v) = M\varphi(v) = \sum_{u\in V} M_{v,u}\varphi(u) = \sum_{\tau\in\Gamma^t}\tilde{M}_{\sigma,\tau}n_\tau\psi(\tau)$. As this quantity is independent of the choice of v (up to the choice of the representative σ) this implies that the eigenvector φ has $\varphi(u) = \varphi(v)$ as long as $w(u) = w(v)$. Thus we define $\tilde{\varphi} \in \mathbb{R}^{k^t}$ by $\tilde{\varphi}(\sigma) = \sqrt{n_\sigma}\psi(\sigma)$ and we have for $w(v) = \sigma$

$$\begin{aligned}
\tilde{M}\tilde{\varphi}(\sigma) &= \sum_{\tau\in\Gamma^t}\tilde{M}_{\sigma,\tau}\tilde{\varphi}(\tau) \\
&= \sum_{\tau\in\Gamma^t} M_{w^{-1}(\sigma),w^{-1}(\tau)}\sqrt{n_\sigma n_\tau}\sqrt{n_\tau}\psi(\tau) \\
&= \sqrt{n_\sigma}\sum_{\tau\in\Gamma^t} M_{w^{-1}(\sigma),w^{-1}(\tau)}n_\tau\psi(\tau) \\
&= \sqrt{n_\sigma}M\varphi(v) \\
&= \sqrt{n_\sigma}\lambda\psi(\sigma) \\
&= \lambda\tilde{\varphi}(\sigma).
\end{aligned}$$

Therefore, the nonzero eigenvalues of M are the same as the nonzero eigenvalues of $\tilde{M}$, and hence it suffices to consider the spectrum of $\tilde{M}$.

For each $\sigma \in \Gamma^t$ define $q_\sigma = Q^{\otimes t}_{\sigma,\sigma}$, that is, q_σ is the probability that an arbitrary vertex v has $w(v) = \sigma$. Now the expected value of n_σ is $q_\sigma n \geq q^t_{\min}n$, and thus by Chernoff bounds $(1-\epsilon^*)\,q_\sigma n \leq n_\sigma \leq (1+\epsilon^*)q_\sigma n$ with probability at least $1-\frac{\epsilon}{2}$, where

$$\epsilon^* = \sqrt{\frac{3\ln\left(\frac{2k^t}{\epsilon}\right)}{q^t_{\min}n}} \leq \sqrt{\frac{3\ln\left(\frac{2n}{\epsilon}\right)}{q^t_{\min}n}} \leq \frac{1}{2} \tag{2}$$

by condition (1). We note that $\delta = \min_{j\in[n]}\left(\sum_{i=1}^n q_i p_{ij}\right) \leq \sum_{i=1}^n q_i = 1$. Moreover, equality holds only in the case that P is the all ones matrix, in which case the graph is complete and the theorem is trivial. Thus, we may assume $\delta < 1$, and we obtain

$$d_{\min} \geq \frac{1}{2}\delta^t n \geq \frac{1}{2}n\delta^{\frac{\ln\left(6\ln\left(\frac{8n}{\epsilon}\right)\right)-\ln(n)}{\ln(\delta)}} = 3\ln\left(\frac{8n}{\epsilon}\right),$$

and thus with probability at least $1-\epsilon$, the spectrum of $\mathcal{L}(G)$ is controlled by the spectrum of $\tilde{M}$ with a signature near the expected signature. Thus we define

$$\begin{aligned}
\mathcal{M}_{\sigma,\tau} &= \sqrt{q_\sigma n q_\tau n}\frac{P^{\otimes t}_{\sigma,\tau}}{\sqrt{\sum_{\eta\in\Gamma^t} nq_\eta P^{\otimes t}_{\sigma,\eta}\sum_{\nu\in\Gamma^t} nq_\nu P^{\otimes t}_{\tau,\nu}}} \\
&= \sqrt{q_\sigma q_\tau}\frac{P^{\otimes t}_{\sigma,\tau}}{\sqrt{\sum_{\eta\in\Gamma^t} q_\eta P^{\otimes t}_{\sigma,\eta}\sum_{\nu\in\Gamma^t} q_\nu P^{\otimes t}_{\tau,\nu}}} \\
&= \sqrt{Q^{\otimes t}_{\sigma,\sigma}Q^{\otimes t}_{\tau,\tau}}\frac{P^{\otimes t}_{\sigma,\tau}}{\sqrt{\sum_{\eta\in\Gamma^t} Q^{\otimes t}_{\eta,\eta}P^{\otimes t}_{\sigma,\eta}\sum_{\nu\in\Gamma^t} Q^{\otimes t}_{\nu,\nu}P^{\otimes t}_{\tau,\nu}}} \\
&= \sqrt{Q^{\otimes t}_{\sigma,\sigma}Q^{\otimes t}_{\tau,\tau}}\frac{P^{\otimes t}_{\sigma,\tau}}{\sqrt{D^{\otimes t}_{\sigma,\sigma}D^{\otimes t}_{\tau,\tau}}} \\
&= \left(D^{-1/2}Q^{1/2}PQ^{1/2}D^{-1/2}\right)^{\otimes t}_{\sigma,\tau},
\end{aligned}$$

where the last two equalities come from the fact that both Q and D are diagonal matrices.

We make the standard observation that for any matrix A and invertible matrix S, the spectrum of $S^{-1}AS^{-1}$ is the same as the spectrum of $S^{-2}A$ as the eigenpairs (λ, v) for $S^{-1}AS^{-1}$ correspond to the eigenpairs $(\lambda, S^{-1}v)$ for $S^{-2}A$. In particular $\|\mathcal{M}\| = \left\|D^{-1/2}Q^{1/2}PQ^{1/2}D^{-1/2}\right\| = \left\|D^{-1}QP\right\| = 1$, as D is formed from the column sums of QP. Now,

$$\begin{aligned}\left\|\tilde{M}-\mathcal{M}\right\| &= \max_{\|f\|=1}\left|f^T\left(\tilde{M}-\mathcal{M}\right)f\right| \\ &= \max_{\|f\|=1}\left|\sum_{\sigma\in\Gamma^t}\sum_{\tau\in\Gamma^t} f_\sigma\left(\tilde{M}-\mathcal{M}\right)_{\sigma,\tau} f_\tau\right| \\ &\leq \max_{\|f\|=1}\sum_{\sigma\in\Gamma^t}\sum_{\tau\in\Gamma^t}|f_\sigma|\left|\tilde{M}-\mathcal{M}\right|_{\sigma,\tau}|f_\tau|.\end{aligned}$$

Notice that from the Chernoff bound on n_σ above, we have that

$$\frac{1+\epsilon^*}{1-\epsilon^*}\mathcal{M}_{\sigma,\tau} \geq \tilde{M}_{\sigma,\tau} \geq \frac{1-\epsilon^*}{1+\epsilon^*}\mathcal{M}_{\sigma,\tau}.$$

Subtracting $\mathcal{M}_{\sigma,\tau}$ from this inequality yields

$$\frac{2\epsilon^*}{1-\epsilon^*}\mathcal{M}_{\sigma,\tau} \geq \tilde{M}_{\sigma,\tau}-\mathcal{M}_{\sigma,\tau} \geq \frac{-2\epsilon^*}{1+\epsilon^*}\mathcal{M}_{\sigma,\tau},$$

and thus $\left|\tilde{M}_{\sigma,\tau}-\mathcal{M}_{\sigma,\tau}\right| \leq \frac{2\epsilon^*}{1-\epsilon^*}\left|\mathcal{M}_{\sigma,\tau}\right|$ for all σ,τ. Moreover, by definition $\mathcal{M}_{\sigma,\tau}$ is positive for all σ,τ. Therefore,

$$\begin{aligned}\left\|\tilde{M}-\mathcal{M}\right\| &\leq \frac{2\epsilon^*}{1-\epsilon^*}\max_{\|f\|=1}\sum_{\sigma\in\Gamma^t}\sum_{\tau\in\Gamma^t}|f_\sigma|\,\mathcal{M}_{\sigma,\tau}\,|f_\tau| \\ &\leq \frac{2\epsilon^*}{1-\epsilon^*}\|\mathcal{M}\| \\ &= \frac{2\epsilon^*}{1-\epsilon^*}\end{aligned}$$

Thus, by Weyl's theorem (see for instance [23]), for any i, $\left|\lambda_i(\tilde{M})-\lambda_i(\mathcal{M})\right| \leq \frac{2\epsilon^*}{1-\epsilon^*} \leq 4\epsilon^*$ by (2). The result for the non-zero eigenvalues follows by the triangle inequality. $\square$

In an analogous manner to Theorem 4 we can control the spectrum of the adjacency matrix of a multiplicative attribute graph.

Theorem 5. *Let G be a t^{th}-order multiplicative attribute graph on n vertices over the alphabet Γ of size k, with probability matrix Q and affinity matrix P. Let D be the diagonal matrix of column sums of QP and let $\Delta < 1$ be the maximum diagonal entry of D and $q_{\min}$ be the minimum diagonal entry of Q. Let $\epsilon > 0$. If*

$$t \leq \min\left\{\frac{\ln(n)-\ln\left(\frac{8}{9}\ln\left(\frac{4n}{\epsilon}\right)\right)}{\ln\left(\frac{1}{\Delta}\right)}, \frac{\ln(n)-\ln\left(12\ln\left(\frac{2n}{\epsilon}\right)\right)}{\ln\left(\frac{1}{q_{\min}}\right)}\right\}, \tag{3}$$

then with probability at least $1-\epsilon$ there is a set $A \subset [n]$, where $A = \{a_1,\ldots,a_{k^t}\}$, such that for $i \in [k^t]$,

$$\left|\lambda_{a_i}(A(G))-\lambda_i\left(\left(Q^{1/2}PQ^{1/2}\right)^{\otimes t}\right)\right| \leq \sqrt{6\Delta^t n\ln\left(\frac{4n}{\epsilon}\right)}+\sqrt{\frac{3\ln\left(\frac{2n}{\epsilon}\right)}{q_{\min}^t n}}.$$

Furthermore, for all $j \notin A$, $|\lambda_j(A(G))| \leq \sqrt{6\Delta^t n\ln\left(\frac{4n}{\epsilon}\right)}$.

As above the bound on t can be transformed into a bound on n in the case that $t\to\infty$, specifically

$$n \geq \max\left\{\frac{8\ln\left(\frac{32}{9\epsilon\Delta^t}\right)}{9\Delta^t}, \frac{12\ln\left(\frac{24}{\epsilon q_{\min}^t}\right)}{q_{\min}^t}\right\}+o(1).$$

Again, we note that there is some ρ, depending only on $\Delta, q_{\min}$, and ϵ, such that if $t \leq \rho\ln(n)$ and n is sufficiently large the conditions for the theorem hold.

5. Applications and Summary

We conclude with some applications of the spectral results for the t^{th}-order stochastic Kronecker graph and multiplicative attribute graph with generating matrix $\mathcal{P} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ and probability matrix $Q = \begin{bmatrix} \mu & 0 \\ 0 & 1-\mu \end{bmatrix}$ for the multiplicative attribute graph.

Theorem 6. *Let G be a t^{th}-order stochastic Kronecker graph with generating matrix $\mathcal{P} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ with $\alpha \geq \gamma$, and $\alpha, \beta, \gamma \in [0,1]$. If $\gamma + \beta > 1$ then for any fixed constant $c > 0$, the following hold with probability at least $1 - 2^{-ct}$:*

(1) *The clustering coefficient of G is*

$$\left(\frac{\alpha^3 + 3\alpha\beta^2 + 3\beta^2\gamma + \gamma^3}{\alpha^2 + \gamma^2 + 2\beta^2 + 2\beta(\alpha+\gamma)}\right)^t + o(1)\,.$$

Furthermore this is asymptotically 0 unless $\alpha = \beta = \gamma = 1$.

(2) *The diameter of G satisfies*

$$\operatorname{diam}(G) \leq 1 + o(1) + \begin{cases} \frac{\ln(4)}{\ln(\beta+\gamma)} & \alpha\gamma - \beta^2 > 0, \gamma \leq \beta \\ \frac{\ln(4)}{\ln(2\beta)} & \alpha\gamma - \beta^2 > 0, \gamma \geq \beta > \frac{1}{2} \\ \frac{\ln(2)}{\ln\left(\frac{2(\alpha+\beta)(\beta+\gamma)}{\alpha\gamma-\beta^2} - 1\right)} t & \alpha\gamma - \beta^2 > 0, \beta < \gamma, \frac{1}{2} \\ \frac{\ln(4)}{\ln(\beta+\gamma)} & \alpha\gamma - \beta^2 = 0 \\ \frac{\ln(4)}{\ln(\gamma+\sqrt{\alpha\gamma})} & \alpha\gamma - \beta^2 < 0, \gamma + \sqrt{\alpha\gamma} > 1 \\ \frac{\ln(2)}{\ln\left(1 + \frac{2(\alpha(\beta+\gamma)+\gamma(\alpha+\beta))}{\beta^2 - \alpha\gamma}\right)} t & \alpha\gamma - \beta^2 < 0, \gamma + \sqrt{\alpha\gamma} \leq 1 \end{cases}.$$

(3) *The chromatic number of G is at most $1 + \left(\frac{\alpha+\gamma+\sqrt{(\alpha-\gamma)^2+4\beta^2}}{2}\right)^t (1+o(1))$. Furthermore, if $\alpha\gamma - \beta^2 < 0$ then the chromatic number satisfies*

$$\frac{2\sqrt{(\alpha-\gamma)^2+4\beta^2}}{\sqrt{(\alpha-\gamma)^2+4\beta^2} - \alpha - \gamma} + o(1) \leq \chi(G) \leq 1 + \left(\frac{\alpha+\gamma+\sqrt{(\alpha-\gamma)^2+4\beta^2}}{2}\right)^t (1+o(1)).$$

(4) *For any two subsets of vertices X and Y of G, the number of edges between them satisfies*

$$\left| e(X,Y) - \frac{\operatorname{vol}(X)\operatorname{vol}(Y)}{\operatorname{vol}(G)} \right| \leq \left(\frac{|\alpha\gamma - \beta^2|}{(\alpha+\beta)(\beta+\gamma)} + o(1) \right) \sqrt{\operatorname{vol}(X)\operatorname{vol}(Y)}.$$

(5) *Let $A = \{(x_i, y_i) \colon x_i \in X, y_i \in Y\}$ be any assignment of the vertices such that each vertex v appears in X and Y with multiplicity $\deg(v)$. If $\alpha\gamma - \beta^2 \geq 0$, then there are paths P_i joining x_i and y_i each of length at most $\frac{2\ln(2)(\alpha+\beta)(\beta+\gamma)}{\beta(\alpha+2\beta+\gamma)} t + o(1)$ such that each edge is contained in at most $\frac{20\ln(2)(\alpha+\beta)(\beta+\gamma)}{\beta(\alpha+2\beta+\gamma)} t + o(1)$ paths. If $\alpha\gamma - \beta^2 < 0$, then there are paths P_i joining x_i and y_i each of length at most $\frac{2\ln(2)(\alpha+\beta)(\beta+\gamma)}{\beta(\alpha+2\beta+\gamma)\left(\frac{\alpha}{\alpha+\beta}+\frac{\gamma}{\gamma+\beta}\right)} t + o(1)$ such that each edge is contained in at most $\frac{20\ln(2)(\alpha+\beta)(\beta+\gamma)}{\beta(\alpha+2\beta+\gamma)\left(\frac{\alpha}{\alpha+\beta}+\frac{\gamma}{\gamma+\beta}\right)} t + o(1)$ paths.*

Proof. We first consider the eigenvalues of the adjacency matrix and note that a quick calculation yields that the eigenvalues of $\mathcal{P}$ are $x \pm y$ where $x = \frac{\alpha+\gamma}{2}$ and $y = \frac{\sqrt{(\alpha-\gamma)^2+4\beta^2}}{2}$. Since $\alpha + \beta \geq \beta + \gamma$ by assumption, Theorem 3 gives that with probability at least $1 - 2^{ct}$ the eigenvalues of the adjacency matrix are such that for each i there are $\binom{t}{i}$ eigenvalues satisfying

$$(x+y)^i (x-y)^{t-i} + \mathcal{O}\left(\sqrt{t(\alpha+\beta)^t}\right).$$

Thus the number of closed walks of length 3 in G is at most

$$\begin{aligned}
\sum_{j=1}^{2^t} \mu_j^3 &= \sum_{i=0}^{t} \binom{t}{i} \left((x+y)^i (x-y)^{t-i} + \mathcal{O}\Big(\sqrt{t(\alpha+\beta)^t} \Big) \right)^3 \\
&= \sum_{i=0}^{t} \binom{t}{i} (x+y)^{3i}(x-y)^{3t-3i} + \sum_{i=0}^{t} \binom{t}{i} (x+y)^{2i}(x-y)^{2t-2i} \mathcal{O}\Big(\sqrt{t(\alpha+\beta)^t} \Big) \\
&\quad + \sum_{i=0}^{t} \binom{t}{i} (x+y)^{i}(x-y)^{t-i} \mathcal{O}\big(t(\alpha+\beta)^t \big) + \sum_{i=0}^{t} \binom{t}{i} \mathcal{O}\Big(\sqrt{t^3(\alpha+\beta)^{3t}} \Big) \\
&= \left((x+y)^3 + (x-y)^3 \right)^t + \left((x+y)^2 + (x-y)^2 \right)^t \mathcal{O}\Big(\sqrt{t(\alpha+\beta)^t} \Big) \\
&\quad + \left((x+y) + (x-y) \right)^t \mathcal{O}\big(t(\alpha+\beta)^t \big) + \mathcal{O}\Big(2^t \sqrt{t^3(\alpha+\beta)^{3t}} \Big) \\
&= \left(2x^3 + 6xy^2 \right)^t + \left(2x^2 + 2y^2 \right)^t \mathcal{O}\Big(\sqrt{t(\alpha+\beta)^t} \Big) + (2x)^t \, \mathcal{O}\big(t(\alpha+\beta)^t \big) + \mathcal{O}\Big(2^t \sqrt{t^3(\alpha+\beta)^{3t}} \Big) .
\end{aligned}$$

Now we note that

$$2x(\alpha+\beta) = (\alpha+\gamma)(\alpha+\beta) \leq 2\alpha(\alpha+\beta) \leq 2(\alpha+\beta)\sqrt{\alpha+\beta},$$

and so $(2x)^t \mathcal{O}(t(\alpha+\beta)^t) \in o\Big(\sqrt{t^3(\alpha+\beta)^{3t}} \Big)$. In a similar manner we have that

$$2x^2 + 2y^2 = \frac{1}{2}\left((\alpha+\gamma)^2 + (\alpha-\gamma)^2 + 4\beta^2 \right) = \frac{1}{2}\left(2\alpha^2 + 2\gamma^2 + 4\beta^2 \right) \leq 2\alpha^2 + 2\beta^2 \leq 2(\alpha+\beta),$$

and so $(2x^2+2y^2)\mathcal{O}\Big(\sqrt{t(\alpha+\beta)^t} \Big) \in o\Big(2^t \sqrt{t^3(\alpha+\beta)^{3t}} \Big)$. Observing that

$$2x^3 + 6xy^2 = \frac{\alpha+\gamma}{4}\left((\alpha+\gamma)^2 + 3(\alpha-\gamma)^2 + 12\beta^2 \right) = \alpha^3 + 3\alpha\beta^2 + 3\beta^2\gamma + \gamma^3$$

we have that the number of closed walks of length 3 in G is $\left(\alpha^3 + 3\alpha\beta^2 + 3\beta^2\gamma + \gamma^3 \right)^t + \mathcal{O}\Big(2^t \sqrt{t^3(\alpha+\beta)^{3t}} \Big)$.

Applying the Chernoff bounds to the degrees, we have $\sum_{j=1}^{2^t} \binom{\deg(v_j)}{2} = \frac{1}{2}\left((\alpha+\beta)^2 + (\beta+\gamma)^2 \right)^t (1+o(1))$. Noting that $\frac{(\alpha+\beta)^2+(\beta+\gamma)^2}{(\alpha+\beta)^{\frac{3}{2}}} = \sqrt{\alpha+\beta} + \frac{(\beta+\gamma)^2}{(\alpha+\beta)^{\frac{3}{2}}} > 1$ and that the contribution of the self-loops to the number of closed walks of length 3 and the number of adjacent edges is $\mathcal{O}(\sum_v \deg(v))$, we have that the clustering coefficient of G is

$$\left(\frac{\alpha^3 + 3\alpha\beta^2 + 3\beta^2\gamma + \gamma^3}{\alpha^2 + \gamma^2 + 2\beta^2 + 2\beta(\alpha+\gamma)} \right)^t + o(1)\,,$$

as desired. Observing that $\alpha^3 + 3\alpha\beta^2 \leq \alpha^2 + 2\alpha\beta + \beta^2$ and $\gamma^3 + 3\gamma\beta^2 \leq \gamma^2 + 2\gamma\beta + \beta^2$ with equality holding simultaneously if and only if $\alpha = \beta = \gamma = 1$, completes the proof of (1).

We now consider the asymptotic behavior of the largest eigenvalue of the adjacency matrix, and to that end consider the function

$$f(\alpha,\gamma) = (x+y) - \sqrt{\alpha+\beta} = \frac{\alpha+\gamma}{2} + \frac{\sqrt{(\alpha-\gamma)^2 + 4\beta^2}}{2} - \sqrt{\alpha+\beta},$$

where β is held constant and restricted to the domain $\gamma \leq \alpha \leq 1$ and $1-\beta < \gamma$. We note that

$$\nabla f(\alpha,\gamma) = \begin{bmatrix} \frac{1}{2} + \frac{\alpha-\gamma}{2\sqrt{(\alpha-\gamma)^2+4\beta^2}} - \frac{1}{2\sqrt{\alpha+\beta}} \\ \frac{1}{2} - \frac{\alpha-\gamma}{2\sqrt{(\alpha-\gamma)^2+4\beta^2}} \end{bmatrix}.$$

In order for $\frac{\partial}{\partial\gamma} f(\alpha,\gamma) = 0$, we must have $1 = \frac{\alpha-\gamma}{\sqrt{(\alpha-\gamma)^2+4\beta^2}}$, but then as $\sqrt{\alpha+\beta} \geq 1$, this implies that $\frac{\partial}{\partial\alpha} f(\alpha,\gamma) \neq 0$. Thus the minimum (and the maximum) value for f must occur along the boundaries given by $\alpha = \gamma$, $\alpha = 1$, or $\gamma = 1-\beta$. We note that $f(\alpha, 1-\beta) = \frac{1+\alpha-\beta}{2} + \frac{1}{2}\sqrt{(\alpha+\beta-1)^2 + 4\beta^2}$ which is an increasing function of α, so $f(\alpha,1-\beta) \geq f(1-\beta,1-\beta) = 0$. Furthermore $f(\alpha,\alpha) = \alpha+\beta-\sqrt{\alpha+\beta} \geq 0$. Finally we observe that $\frac{\partial}{\partial\gamma} f(1,\gamma) = \frac{1}{2} - \frac{1-\gamma}{2\sqrt{(1-\gamma)^2+4\beta^2}} \geq 0$ and so $f(1,\gamma) \geq f(1,1-\beta) \geq f(1-\beta,1-\beta) = 0$. Thus $f(\alpha,\gamma) \geq 0$ for all feasible choices of α,β,γ and furthermore $f(\alpha,\gamma) > 0$ if $\gamma+\beta > 1$. Thus we have that

$\mathcal{O}\left(\sqrt{t(\alpha+\beta)^t}\right) \in o((x+y)^t)$ and so the largest eigenvalue of the adjacency matrix of G is $(x+y)^t(1+o(1))$. Furthermore if $x<y$ then the smallest eigenvalue is $(x+y)^{t-1}(x-y)(1+o(1))$. Observing that $x<y$ if and only if $\alpha\gamma-\beta^2<0$ and applying Hoffman's [21, 22] and Wilf's [43] bounds give result (3).

We now turn our attention to the spectrum of the normalized Laplacian of G and note that by Theorem 3 the relevant matrix for the spectrum is $\begin{bmatrix} \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\sqrt{(\alpha+\beta)(\beta+\gamma)}} \\ \frac{\beta}{\sqrt{(\alpha+\beta)(\beta+\gamma)}} & \frac{\gamma}{\beta+\gamma} \end{bmatrix}$. The eigenvalues of this matrix are 1 and $\frac{\alpha\gamma-\beta^2}{(\alpha+\beta)(\beta+\gamma)}$ and thus the second smallest eigenvalue of the normalized Laplacian of G is

$$1-\frac{\alpha\gamma-\beta^2}{(\alpha+\beta)(\beta+\gamma)}+o(1)=\beta\frac{\alpha+2\beta+\gamma}{(\alpha+\beta)(\beta+\gamma)}+o(1)$$

if $\alpha\gamma-\beta^2\geq 0$ and

$$1-\frac{(\alpha\gamma-\beta^2)^2}{(\alpha+\beta)^2(\beta+\gamma)^2}+o(1)=\beta\frac{\alpha+2\beta+\gamma}{(\alpha+\beta)(\beta+\gamma)}\left(\frac{\alpha}{\alpha+\beta}+\frac{\gamma}{\beta+\gamma}\right)+o(1)$$

otherwise. Together these eigenvalues bound the deviation from 1 of the non-principle eigenvalue and yield the discrepancy result (4). These bounds also immediately yield the result on the congestion of paths in G.

We now consider the diameter of G in the case when $\alpha\gamma-\beta^2=0$. In this case all of the non-principle eigenvalues of $\mathcal{L}$ are centered around 1 and hence we have that

$$1-3\sqrt{\frac{3\ln(2^{(1+c)t+2})}{(\beta+\gamma)^t}}\leq\lambda_2\leq\lambda_n\leq 1+3\sqrt{\frac{3\ln(2^{(1+c)t+2})}{(\beta+\gamma)^t}}.$$

Thus we have that

$$\frac{\lambda_n+\lambda_2}{\lambda_n-\lambda_2}=1+\frac{2\lambda_2}{\lambda_n-\lambda_2}\geq 1+\frac{2\left(1-3\sqrt{\frac{3\ln(2^{(1+c)t+2})}{(\beta+\gamma)^t}}\right)}{6\sqrt{\frac{3\ln(2^{(1+c)t+2})}{(\beta+\gamma)^t}}}=\frac{1}{3\sqrt{\frac{3\ln(2^{(1+c)t+2})}{(\beta+\gamma)^t}}}=\sqrt{\frac{(\beta+\gamma)^t}{\ln\left(2^{27(1+c)t+54}\right)}}.$$

Thus we have that

$$\operatorname{diam}(G)\leq\left\lceil\frac{\ln(2^t-1)}{\ln\left(\frac{\lambda_n+\lambda_2}{\lambda_n-\lambda_2}\right)}\right\rceil\leq\left\lceil\frac{t\ln(2)}{\frac{1}{2}t\ln(\beta+\gamma)-\frac{1}{2}\ln\ln\left(2^{27(1+c)t+54}\right)}\right\rceil=\left\lceil\frac{2\ln(2)}{\ln(\beta+\gamma)-o(1)}\right\rceil.$$

Now if $\alpha\gamma-\beta^2\neq 0$, we note that if H is a t^{th}-order Stochastic Kronecker graph generated by $\mathcal{P}'$ and if $\mathcal{P}'$ is dominated by $\mathcal{P}$ component-wise, then by the natural coupling we have that $\operatorname{diam}(G)\leq\operatorname{diam}(H)$. Thus we consider the following matrices

$$\mathcal{P}'=\begin{bmatrix}\alpha' & \beta' \\ \beta' & \gamma'\end{bmatrix}=\begin{cases}\begin{bmatrix}\frac{\beta^2}{\gamma} & \beta \\ \beta & \gamma\end{bmatrix} & \alpha\gamma-\beta^2>0, \gamma\leq\beta \\ \begin{bmatrix}\beta & \beta \\ \beta & \beta\end{bmatrix} & \alpha\gamma-\beta^2>0, \gamma>\beta>\frac{1}{2} \\ \begin{bmatrix}\alpha & \sqrt{\alpha\gamma} \\ \sqrt{\alpha\gamma} & \gamma\end{bmatrix} & \alpha\gamma-\beta^2<0, \gamma+\sqrt{\alpha\gamma}>1\end{cases}.$$

It is easy to verify that in all these cases, $\mathcal{P}$ dominates $\mathcal{P}'$ component-wise, $\alpha'\geq\gamma'$ and $\beta'+\gamma'>1$. The remaining cases yield a linear upper bound, and complete the argument for (2). □

We note that portions of the result (2) were proven previously by Mahdian and Xu [31] who analyzed the case where $\alpha\geq\beta\geq\gamma$. In their work they claim a bound on the diameter of $2+\frac{1}{\beta+\gamma-1}$ which roughly agrees with the bound $\left\lceil\frac{\ln(4)}{\ln(\beta+\gamma)}\right\rceil+o(1)$. This result represents partial progress towards their conjecture that any stochastic Kronecker graph with generating matrix such that $\alpha+\beta>1$ and $\beta+\gamma>1$ has constant diameter with high probability. We believe that essential obstructions to confirming their conjecture are the generating matrices

$$\begin{bmatrix} 1 & \epsilon \\ \epsilon & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \epsilon & 1 \\ 1 & \epsilon \end{bmatrix}.$$

Before considering the implications of the spectrum for the multiplicative attribute graph, we note that the spectrum of $Q^{1/2}\mathcal{P}Q^{1/2}$ is $x \pm y$ where $x = \frac{\mu\alpha + (1-\mu)\gamma}{2} = \frac{\Delta + \delta - \beta}{2}$ and $y = \frac{1}{2}\sqrt{(\mu\alpha - (1-\mu)\gamma)^2 + 4\mu(1-\mu)\beta^2} = \frac{1}{2}\sqrt{(\Delta - \delta)^2 + (4\mu - 2)\beta(\Delta - \delta) + \beta^2}$, where $\Delta = \mu\alpha + (1-\mu)\beta \geq \mu\beta + (1-\mu)\gamma = \delta$. Viewing $2x + 2y$ as a function f of β with δ, Δ, μ fixed, we have

$$f'(\beta) = -1 + \frac{2\beta + (4\mu - 2)(\Delta - \delta)}{2\sqrt{(\Delta - \delta)^2 + (4\mu - 2)\beta(\Delta - \delta) + \beta^2}}$$

We find that there is some β where $f'(\beta) = 0$ if and only if $(4\mu - 2)^2(\Delta - \delta)^2 = 4(\Delta - \delta)^2$, which occurs only in the degenerate cases where $\mu = 0$ or $\mu = 1$. Furthermore, $f'(0) = -1 + \frac{1}{2}(4\mu - 2) = -2 + 2\mu < 0$. Thus for all $\beta \in [0, 1]$, $f(\beta) \leq f(0) = \Delta < 1$. Thus by Theorem 5, the eigenvalues of the adjacency matrix are $\mathcal{O}\left(\sqrt{\Delta^t n \ln\left(\frac{n}{\epsilon}\right)}\right)$, with the spectrum of $Q^{\frac{1}{2}}PQ^{\frac{1}{2}}$ contributing essentially no information to the spectrum of the multiplicative attribute graph. However, with appropriate controls on t, we see from Theorem 4 that the spectrum of the normalized Laplacian can be controlled. Specifically, using similar techniques as Theorem 6, we have the following theorem.

Theorem 7. *Let G be a t^{th}-order multiplicative attribute graph on n vertices with generating matrix $\mathcal{P} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$, probability matrix $Q = \begin{bmatrix} \mu & 0 \\ 0 & 1-\mu \end{bmatrix}$ with $\mu\alpha + (1-\mu)\beta \geq \mu\beta + (1-\mu)\gamma$, and $\alpha, \beta, \gamma, \mu \in (0, 1)$. Let $\Delta = \mu\alpha + (1-\mu)\beta$ and $\delta = \mu\beta + (1-\mu)\gamma$ and let ρ be a fixed constant such that $\frac{t}{\log(n)} \leq \rho$. For any fixed constant $c > 0$ and for sufficiently large n, with probability at least $1 - n^{-c}$ the graph G satisfies:*

(1) *If $\rho \leq \frac{-1}{\ln(\Delta q_{\min}^2)}$, then the clustering coefficient of G is asymptotically 0.*

(2) *If $\rho < \frac{-1}{\ln(\delta)}, \frac{-1}{\ln(q_{\min})}$ then the diameter satisfies*

$$\mathrm{diam}(G) \leq 1 + o(1) + \begin{cases} \frac{2\ln(n)}{1 - \rho\min\{\ln(\delta), \ln(q_{\min})\}} & \alpha\gamma - \beta^2 > 0, \gamma \leq \beta \\ \frac{\ln(n)}{\ln\left(1 + 2\frac{(\delta\Delta)^{t-1} - \mu(1-\mu)(\alpha\gamma - \beta^2)}{(\delta\Delta)^{t-1}\mu(1-\mu)(\alpha\gamma-\beta^2) - \left(\mu(1-\mu)(\alpha\gamma-\beta^2)\right)^t}\right)} & \alpha\gamma - \beta^2 > 0, \beta < \gamma \\ \frac{2\ln(n)}{1 - \rho\min\{\ln(\delta), \ln(q_{\min})\}} & \alpha\gamma - \beta^2 = 0 \\ \frac{\ln(n)}{\ln\left(1 + 2\frac{(\delta\Delta) - \mu(1-\mu)(\alpha\gamma-\beta^2)}{(\delta\Delta)\mu(1-\mu)(\alpha\gamma-\beta^2) - \left(\mu(1-\mu)(\alpha\gamma-\beta^2)\right)^2}\right)} & \alpha\gamma - \beta^2 < 0 \end{cases}.$$

(3) *If $\rho < \frac{-1}{\ln(\Delta)}, \frac{-1}{\ln(q_{\min})}$, then the chromatic number of G is at most $1 + \left(\sqrt{6(1+c)\Delta^t n \ln(n)}\right)(1 + o(1))$.*

(4) *For any two subsets of vertices X and Y of G, the number of edges between them satisfies*

$$\left| e(X, Y) - \frac{\mathrm{vol}\,(X)\,\mathrm{vol}\,(Y)}{\mathrm{vol}\,(G)} \right| \leq \left(\frac{\mu(1-\mu)\left|\alpha\gamma - \beta^2\right|}{\Delta\delta} + o(1) \right) \sqrt{\mathrm{vol}\,(X)\,\mathrm{vol}\,(Y)}.$$

(5) *Let $A = \{(x_i, y_i) \colon x_i \in X, y_i \in Y\}$ be any assignment of the vertices such that each vertex v appears in X and Y with multiplicity $\deg(v)$. If $\alpha\gamma - \beta^2 \geq 0$, then there are paths P_i joining x_i and y_i each of length at most $\frac{2\ln(n)\Delta\delta}{\beta(\mu^2\alpha + 2\mu(1-\mu)\beta + (1-\mu)^2\gamma)} + o(1)$ such that each edge is contained in at most $\frac{20\ln(n)\Delta\delta}{\beta(\mu^2\alpha + 2\mu(1-\mu)\beta + (1-\mu)^2\gamma)} + o(1)$ paths. If $\alpha\gamma - \beta^2 < 0$, then there are paths P_i joining x_i and y_i each of length at most $\frac{2\ln(n)\Delta\delta}{\beta(\mu^2\alpha + 2\mu(1-\mu)\beta + (1-\mu)^2\gamma)\left(\frac{\mu\alpha}{\Delta} + \frac{(1-\mu)\gamma}{\delta}\right)} + o(1)$ such that each edge is contained in at most $\frac{20\ln(n)\Delta\delta}{\beta(\mu^2\alpha + 2\mu(1-\mu)\beta + (1-\mu)^2\gamma)\left(\frac{\mu\alpha}{\Delta} + \frac{(1-\mu)\gamma}{\delta}\right)} + o(1)$ paths.*

From these theorems, particular with regards to the diameter, it is clear that the situation where the generating matrix has a zero determinant, that is $\alpha\gamma - \beta^2 = 0$, is a particularly well behaved class of stochastic Kronecker and multiplicative attribute graphs. More specifically, the spectrum of the normalized Laplacian for both of these graphs converges with t to a point mass at 1.

We consider this special regime of the stochastic Kronecker and note that if v is a vertex with weight k, then

$$\mathbb{E}\left[\deg(v)\right] = (\alpha+\beta)^k(\beta+\gamma)^{t-k} = \left(\frac{\beta^2}{\gamma}+\beta\right)^k(\beta+\gamma)^{t-k} = \left(\frac{\beta}{\gamma}\right)^k(\beta+\gamma)^t.$$

Moreover, for any $v \in V(G)$ and transposition $\pi \in S_n$, we have that $\mathbb{P}(v \sim u) = \mathbb{P}(v \sim \pi(u))$, as either we exchange two equal coordinates and preserve probabilities, or we exchange unequal coordinates, resulting in either a preservation of probabilities or switching $\alpha\gamma$ for β^2 (or vice versa). Thus the only information that contributes to the probability of adjacency of two vertices u and v is their respective weights. In particular, it can easily be show that

$$\mathbb{P}(u \sim v) = \beta^{|v|+|u|}\gamma^{t-(|v|+|u|)},$$

where $|v|$ is the weight of the vertex v.

On the other hand, we also have that

$$\begin{aligned}
\frac{\mathbb{E}\left[\deg(v)\right]\mathbb{E}\left[\deg(u)\right]}{\sum_w \mathbb{E}\left[\deg(w)\right]} &= \frac{\left(\frac{\beta}{\gamma}\right)^{|v|}(\beta+\gamma)^t\left(\frac{\beta}{\gamma}\right)^{|u|}(\beta+\gamma)^t}{(\beta+\gamma)^t\left(\frac{\beta}{\gamma}+1\right)^t} \\
&= \gamma^t\left(\frac{\beta}{\gamma}\right)^{|v|+|u|} \\
&= \beta^{|v|+|u|}\gamma^{t-(|v|+|u|)} = \mathbb{P}(u \sim v).
\end{aligned}$$

Thus the 2×2 stochastic Kronecker where the generating matrix has determinant zero is a special case of a $G(\vec{w})$ graph, a random graph with a given expected degree sequence and independent edges. Such graphs have been studied extensively in, for example, [9, 10, 12, 13]. Furthermore, although the multiplicative attribute graph in this case is not an expected degree sequence graph, it is a blow-up of such a graphs and thus unsurprisingly preserves many of the same properties.

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