Discrete Math

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Day R(4,4)-4
The Pigeonhole Principle: If $n + 1$ pigeons are placed into $n$ holes, then one hole has at least two pigeons.
Claim: In any collection of 6 people, there are always 3 mutual acquaintances or three mutual strangers.
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That’s right, this is a graph theory problem: we can restate it as “In any graph with 6 vertices, there are always three mutually adjacent vertices or three mutually nonadjacent vertices.”
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**Claim, Again:** Let $G$ be a complete graph on 6 vertices. If we color the edges of $G$ red or blue, then we can either find a clique with three vertices whose edges are all red, or a clique with three vertices whose edges are all blue.

Furthermore, this isn't true if $G$ only has 5 vertices.
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Theorem: (Ramsey, 1930) For $k, l_1, \ldots, l_k \in \mathbb{N}$, there is a number $N$ such that if we color the edges of $K_N$ with the “colors” 1, 2, \ldots, $k$, then we can always find a clique with $l_i$ vertices whose edges all have color $i$, for some $i$. ($i$ depends on the actual coloring.)
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Notation: The Ramsey Number $R(l_1, \ldots, l_k)$ is the smallest value of $N$ that satisfies Ramsey’s Theorem for the given $k, l_1, \ldots, l_k$.
Ramsey’s Theorem: A Proof When $k = 2$.

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We can show that if $s, t > 2$, and $R(s - 1, t)$ and $R(s, t - 1)$ exist, then by letting

$$N = R(s - 1, t) + R(s, t - 1)$$

and coloring $K_N$ with two colors, we’ll always find either a clique with $s$ vertices whose edges are red or a clique with $t$ vertices whose edges are blue.
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and coloring $K_N$ with two colors, we’ll always find either a clique with $s$ vertices whose edges are red or a clique with $t$ vertices whose edges are blue. In particular, this means

$$R(s, t) \leq R(s - 1, t) + R(s, t - 1).$$
The relationship between Ramsey’s Theorem and the Pigeonhole Principle isn’t hard to guess, especially if we restate the latter:

**The Pigeonhole Principle, Take 2:** If $n + 1$ objects are colored with $n$ colors, then at least two objects have the same color.
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**The Pigeonhole Principle, Take 2:** If \( n + 1 \) objects are colored with \( n \) colors, then at least two objects have the same color.

Ramsey’s Theorem gives a much stronger variant: not only do we get several edges with the same color, we actually guarantee a more complicated structure exists!
Some Known Ramsey Numbers.

We saw $R(3, 3) = 6$, and it turns out $R(4, 4) = 18$. Additionally, the numbers $R(3, k)$ with $4 \leq k \leq 9$, $R(4, 5)$, and $R(3, 3, 3)$ are known.
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and, skipping ahead,

$$798 \leq R(10, 10) \leq 23556.$$