A SUBSTITUTE FOR SATURATION IN ABSTRACT ELEMENTARY CLASSES

MONICA VANDIEREN

Abstract. This paper continues work of Shelah and Villaveces from [ShVi]. Shelah and Villaveces work towards a downward categoricity transfer theorem in abstract elementary classes (AEC) with no maximal models under GCH and a version of the weak diamond.

One of the main conjectures in [ShVi] was the uniqueness of limit models. Here we solve this problem under an additional assumption. Suppose that K is an AEC categorical in some λ, above the Hanf number and that for $\mu < \lambda$, K^{am}_μ is closed under unions of length $< \mu^+$.

Theorem 0.1 (Uniqueness of Limit Models). *Let θ_1, θ_2 be limit ordinals $< \mu^+$. If M_1 is a (μ, θ_1)-limit model over M and M_2 is a (μ, θ_2)-limit model over M, then M_1 is isomorphic to M_2.*

In order to prove the uniqueness theorem, we develop the theory of Galois-splitting in AECs. In particular we prove the extension property for Galois-splitting in AECs:

Theorem 0.2 (Extension Property for Splitting). *Suppose that $\text{ga-tp}(a/M)$ does not μ-split over N and that M is universal over N. For every $M' \in K^{am}_\mu$ with $M \prec_K M'$, we have that there exists $q \in \text{ga-S}(M')$ such that $q \geq \text{ga-tp}(a/M)$ and q does not μ-split over N.*

We generalize Theorem 0.2 and prove the $<^c$-extension property for towers. This is also a partial solution to a problem from [ShVi]. In our proof of Theorem 0.1, we develop the concept of full towers. We show that the union of full towers is full.

1. Introduction

Shelah's paper, [Sh702] is based on a series of lectures given at Rutgers University. In the lectures, Shelah elaborates on open problems in model theory which he has attempted but which have not yet been solved. There Shelah refers to the subject of Section 1.3, "Classification of Non-elementary Classes", as the major problem of model theory. He points out that one of the main steps in classifying non-elementary classes is the development of stability theory. In first order logic, solutions to Łoś' Conjecture produced machinery that advanced the study of stability theory. It is natural, then, to

---

Date: March 25, 2002.

Preliminary Draft This paper is part of the author's Ph.D. thesis written under the guidance of Rami Grossberg.

consider a generalization of this conjecture as a test question for a proposed stability theory for AECs:

Conjecture 1.1 ([Sh702]). If K is an abstract elementary class that is categorical in some $\lambda > Hanf(K)$, then K is categorical in every $\lambda > Hanf(K)$.

Despite the existence of over 500 published pages of partial results towards this conjecture, it remains very open. Since the mid-eighties, model theorists have approached Shelah's conjecture from two different directions. Shelah, M. Makkai and O. Kolman attacked the conjecture with set theoretic assumptions (see [MaSh], [KoSh] and [Sh472]). On the other hand, Shelah also looked at the conjecture under additional model theoretic assumptions in [Sh394] and [Sh600]. More recent work of Shelah and A. Villaveces [ShVi] profits from both model theoretic and set theoretic assumptions, however these assumptions are weaker than the hypotheses made in [MaSh], [KoSh], [Sh472], [Sh394], and [Sh600]. A main feature of their context is that they work in AECs where the amalgamation property is not known to hold. This paper focuses on resolving problems from [ShVi]. Here we recall the context of [ShVi] (Assumptions 1.2(1) through 1.2(5)).

Assumption 1.2. We make the following assumptions for the remainder of the paper:

(1) K is an abstract elementary class,
(2) K has no maximal models,
(3) K is categorical in some $\lambda > LS(K)$,
(4) GCH holds and
(5) $\Phi_{\mu^+}(S^{\mu^+}_\theta)$ holds for every cardinal $\mu < \lambda$, and every regular θ with $\theta < \mu^+$.

Assumption 1.2(5) is not explicitly made in [ShVi]. We believe this version of weak diamond is needed to carry out Shelah and Villaveces's suggestion for the proof that limit models are amalgamation bases. We provide a complete proof of the theorem which uses Assumption 1.2(5) (see Theorem 4.3) and give an exposition of the strength of Assumption 1.2.5 in Section 4.

In 1985 Rami Grossberg made the following conjecture:

Conjecture 1.3. If K is an AEC categorical above the Hanf number, then every $M \in K$ is an amalgamation base.

This conjecture encouraged Shelah to produce a partial solution to the categoricity conjecture under the assumption that every model $M \in K$ is an amalgamation base [Sh394]. This result directs future work towards the categoricity conjecture to solving Conjecture 1.3. The underlying goal of [ShVi] was to make progress towards Conjecture 1.3 under Assumption 1.2. Not knowing that every model is an amalgamation base presents several obstacles in applying known notions and techniques. For instance, there may exist some models over which we cannot even define the most basic notion of a type.

One approach to Conjecture 1.3 is to see if arguments from [KoSh] can be carried out in this more general context. Shelah and Kolman prove Conjecture 1.3 for $L_{\kappa,\omega}$ theories where κ is a measurable cardinal. They first introduce limit models as a substitute for saturated models, and then prove the uniqueness of limit models. A major objective of [ShVi] was to show the uniqueness of limit models:

Conjecture 1.4 (Uniqueness of Limit Models). Suppose Assumption 1.2 holds. For $\theta_1, \theta_2 < \mu^+ < \kappa$, if M_1 and M_2 and (μ,θ_1)-, (μ,θ_2)-limit models over M, respectively, then M_1 is isomorphic to M_2.

While limit models were used to prove that every model is an amalgamation base in [KoSh], limit models played a behind-the-scenes role in Shelah's downward solution to the categoricity conjecture in [Sh394]. Furthermore, there is evidence that the uniqueness of limit models provides a basis for the development of a notion of non-forking for abstract elementary classes. Such a notion could be used in the development of a stability theory for AECs.

In the Fall of 1999, I identified a gap in Shelah and Villaveces' proof of uniqueness of limit models. As of the Fall of 2001, Shelah and Villaveces could not resolve the problem.

This paper includes an exposition of selected results from [ShVi] and includes a partial solution to the uniqueness of limit models (see Theorem 2.28). In order to prove Theorem 2.28, we prove several facts about splitting and full towers which stand alone.

The author would like to thank John Baldwin for his unlimited willingness to discuss and comment on this paper. Additionally the author would like to thank Olivier Lessmann and Andres Villaveces for their comments on preliminary versions of this paper.

2. Background

Definition 2.1. K is an abstract elementary class (AEC) if K is a class of models for some vocabulary L and is equipped with a binary relation, $\preceq_K$ satisfying the following:

(1) Closure under isomorphisms.
(2) $\preceq_K$ refines the submodel relation.
(3) $\preceq_K$ is a partial order on K.
(4) If $\langle M_i \mid i<\delta\rangle$ is a $\prec_K$-increasing chain of models in K
   (a) $\bigcup_{i<\delta} M_i \in K$,
   (b) for every $j<\delta$, $M_j \prec_K \bigcup_{i<\delta} M_i$ and
   (c) if $M_i \prec_K N$ for every $i<\delta$, then $\bigcup_{i<\delta} M_i \prec_K N$.
(5) If $M_0, M_1 \preceq_K N$ and M_0 is a submodel of M_1, then $M_0 \preceq_K M_1$.

(6) (Downard Löwenheim-Skolem Axiom) There is a Löwenheim-Skolem number of K, denoted $LS(K)$ which is the minimal λ such that for every $N \in K$ and every $A \subset N$, there exists M with $A \subseteq M \prec_K N$ of cardinality $\lambda + |A|$.

Notation 2.2. If μ is a cardinal and K is an abstract elementary class, K_μ is the collection of elements of K with cardinality μ.

Definition 2.3. For models M, N in an AEC, K, the mapping $f: M \to N$ is an $\prec_K$-embedding iff f is an injective $L(K)$-homomorphism and $f[M] \preceq_K N$.

Using the axioms of AEC, one can show that Axiom 4 has an alternative formulation (see [Sh88] or Chapter 13 of [Gr]):

Proposition 2.4 (P.M. Cohn 1965). *Let $(I, \leq)$ be a directed set. If $\langle M_t \mid t \in I\rangle$ and $\{h_{t,s} \mid t \leq s \in I\}$ are such that*

(1) *for $t \in I$, $M_t \in K$*
(2) *for $t \leq s \in I$, $h_{t,s}: M_t \to M_s$ is a $\prec_K$-embedding and*
(3) *for $t_1 \leq t_2 \leq t_3 \in I$, $h_{t_1,t_3} = h_{t_2,t_3} \circ h_{t_1,t_2}$ and $h_{t,t} = id_{M_t}$,*

then, whenever $s = \lim_{t \in I} t$, there exist $M_s \in K$ and $\prec_K$-mappings $\{h_{t,s} \mid t \in I\}$ such that

$$h_{t,s}: M_t \to M_s \text{ and}$$
$$\text{for } t_1 \leq t_2 \leq s,\ h_{t_1,s} = h_{t_2,s} \circ h_{t_1,t_2} \text{ and } h_{s,s} = id_{M_s}.$$

Definition 2.5. A partially ordered set $(I, \leq)$ is directed iff for every $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$.

In fact we can conclude more about directed limits (Lemma 2.6). We will use this lemma in our proofs of the extension property for towers.

Lemma 2.6. *Suppose that $\langle M_t \prec_K N_t \mid t \in I\rangle$ and $\{f_{t,s} \mid t \leq s \in I\}$ is a directed system with $f_{t,s}: N_t \to N_s$ and $f_{t,s} \restriction M_t: M_t \to M_s$. If M^* and N^* are direct limits of the directed systems $\langle M_t, f_{t,s} \restriction M_t \mid t \leq s \in I\rangle$ and $\langle N_t, f_{t,s} \mid t \leq s \in I\rangle$, respectively, then $M^* \hookrightarrow_K N^*$ and $f_{t,\sup(I)} \restriction M_t: M_t \to M^*$.*

The proof of Lemma 2.6 is straightforward using the following proposition:

Proposition 2.7 ([Sh88] or see [Gr]). *$K^{\prec_K} := \{(N, M) \mid M, N \in K, M \prec_K N\}$ is an abstract elementary class with $L(K^{\prec_K}) = L(K) \cup \{P\}$ where P is a unary predicate and $\prec_{K^{\prec_K}}$ is defined by*

$$(N, M) \prec_{K^{\prec_K}} (N', M') \iff (N \prec_K N' \text{ and } M \prec_K M').$$

We will use Lemma 2.6 as well as the trivial observation (Claim 2.8) in the proof of the Conjecture 1.4

Claim 2.8. *If $\langle N_t \mid t < s\rangle$ and $\{f_{r,t} \mid r < t < s\}$ form a directed system and for every $r \leq t < s$ we have that $N_t = N_r = N$ and $f_{r,t} \in Aut(N)$. Then a direct limit $(N_s, \langle f_{t,s} \mid t \leq s\rangle)$ of this system is such that $f_{t,s}: N_t \cong N_s$ for every $t \leq s$. Moreover we can choose a direct limit such that $N_s = N$.*

The following gives a characterization of AECs as PC-classes. Theorem 2.10 is often referred to as Shelah's Presentation Theorem.

Definition 2.9. A class K of structures is called PC_Γ class if there exists a language L_1, a first order theory, T_1, in the language L_1, and a collection of types without parameters, Γ, such that L_1 is an expansion of $L(K)$ and

$$K = PC(T_1, \Gamma, L) := \{M \restriction L : M \models T_1 \text{ and } M \text{ omits all types from } \Gamma\}.$$

When $|T_1| + |L_1| + |\Gamma| + \aleph_0 = \lambda$, we say that K is PC_λ.

Theorem 2.10 (Lemma 1.8 of [Sh88] or [Gr]). If $(K, \prec_K)$ is an AEC, then there exists $\lambda \leq 2^{LS(K)}$ such that K is PC_λ.

In Section 3 we will see that this presentation of AECs as PC-classes allows us to construct Ehrenfeucht-Mostowski models.

Definition 2.11. Let K be an abstract elementary class.

(1) Let λ, κ_1 and κ_2 be cardinals with $\lambda \leq \kappa_1, \kappa_2$. We say that $M \in K_\lambda$ is a (κ_1, κ_2)-amalgamation base if for every $N_1 \in K_{\kappa_1}$ and $N_2 \in K_{\kappa_2}$ and $g_i : M \to N_i$ for $(i = 1, 2)$, there are $\prec_K$-embeddings f_i, $(i = 1, 2)$ and a model N such that the following diagram commutes:

$$\begin{tikzcd}
N_1 \arrow[r, "f_1"] & N \\
M \arrow[u, "g_1"] \arrow[r, "g_2"'] & N_2 \arrow[u, "f_2"']
\end{tikzcd}$$

(2) We say that a model $M \in K_\lambda$ is an amalgamation base if M is a (λ, λ)-amalgamation base.
(3) We write K^{am} for the class of amalgamation bases which are in K.
(4) We say K satisfies the amalgamation property iff for every $M \in K$, M is an amalgamation base.

Remark 2.12. We get an equivalent definition of amalgamation base, if we additionally require that $g_i \restriction M = id_M$ for $i = 1, 2$, in the definition above. See [Gr] for details.

Amalgamation bases are central in the definition of types. Since we are not working in a fixed logic, we will not define types as collections of formulas. Instead, we will define types as equivalence classes with respect to images under $\prec_K$-mappings:

Definition 2.13. For triples $(\bar{a}_l, M_l, N_l)$ where $\bar{a}_l \in N_l$ and $M_l \prec_K N_l \in K$ for $l = 0, 1$, we define a binary relation E as follows: $(\bar{a}_0, M_0, N_0) E (\bar{a}_1, M_1, N_1)$ iff $M_0 = M_1$ and there exists $N \in K$ and $\prec_K$-mappings f_0, f_1 such that $f_l : N_l \to N$ and $f_l \restriction M = id_M$ for $l = 0, 1$ and $f_0(\bar{a}_0) = f_1(\bar{a}_1)$.

$$\begin{tikzcd} N_1 \arrow[r, "f_1"] & N \\ M \arrow[u, "id"] \arrow[r, "id"'] & N_2 \arrow[u, "f_2"'] \end{tikzcd}$$

Remark 2.14. E is an equivalence relation on the set of triples of the form $(\bar a, M, N)$ where $M \preceq_K N$, $\bar a \in N$ and $M, N \in K^{am}_\mu$ for fixed $\mu \geq LS(K)$.

In AEC with the amalgamation property, we are often limited to speak of types only over models. Here we are further restricted to deal with types only over models which are amalgamation bases.

Definition 2.15. Let $\mu \geq LS(K)$ be given.

(1) For $M, N \in K^{am}_\mu$ and $\bar a \in {}^{<\omega}N$, the *Galois-type of* $\bar a$ *in* N *over* M, written ga-tp$(\bar a/M, N)$, is defined to be $(\bar a, M, N)/E$.
(2) For $M \in K^{am}_\mu$, ga-$S^1(M) := \{$ga-tp$(a/M, N) \mid M \preceq N \in K^{am}_\mu; a \in N\}$.
(3) We say $p \in$ ga-$S(M)$ is *realized in* N whenever $M \prec_K N$ and there exists $a \in N$ and $N' \in K^{am}_\mu$ such that $p = (a, M, N')/E$.

Remark 2.16. We refer to these types as Galois types to distinguish them from notions of types defined as a collection of formulas.

Proposition 2.17 (see [Gr]). *When* $K = Mod(T)$ *for* T *a complete first order theory, the above definition of* ga-tp$(a/M, N)$ *coincides with the classical first order defintion where* c *and* a *have the same type over* M *iff for every first order formula* $\varphi(x, \bar b)$ *with parameters from* M,

$$\models \varphi(c, \bar b) \leftrightarrow \models \varphi(a, \bar b).$$

Proof. By Robinson's Consistency Theorem. $\dashv$

Definition 2.18. We say that K is *stable in* μ iff for every $M \in K^{am}_\mu$, $|$ga-$S^1(M)| = \mu$.

Fact 2.19 (Fact 2.1.3 of [ShVi]). *Since* K *is categorical in* λ, *for every* $\mu < \lambda$, *we have that* K *is stable in* μ.

Definition 2.20. (1) Let κ be a cardinal. We say N is κ*-universal over* M iff for every $M' \in K_\kappa$ with $M \prec_K M'$ there exists a $\prec_K$-embedding $g : M' \to N$ such that $g \restriction M = id_M$:

$$\begin{tikzcd} M' \arrow[dr, "g"] & \\ M \arrow[u, "id"] \arrow[r, "id"'] & N \end{tikzcd}$$

(2) We say N is *universal over* M iff N is $\|M\|$-universal over M.

Universal extensions exist in many stable contexts [Sh600]. We provide a reference to [ShVi] where a proof is provided from stronger assumptions including categoricity and GCH (see Lemma 2.21).

Lemma 2.21 (Theorem 1.3.1 from [ShVi]). For every μ with $LS(K) < \mu < \lambda$, if $M \in K^{am}_\mu$, then there exists $M' \in K^{am}_\mu$ such that M' is universal over M.

Notice that the following proposition asserts that it is unreasonable to prove a stronger existence statement than Lemma 2.21 without having proved the amalgamation property.

Proposition 2.22. If M' is universal over M, then M is an amalgamation base.

As mentioned in the introduction, limit models were introduced by Kolman and Shelah in [KoSh]. After proving the uniqueness of limit models in their context, Shelah and Kolman derive the Amalgamation Property. The main goal of this paper is to prove the uniqueness of limit models in the context of [ShVi] under an additional assumption.

Definition 2.23. For $M', M \in K_\mu$ and σ a limit ordinal with $\sigma < \mu^+$, we say that M' is a (μ,σ)-limit over M iff there exists a $\prec_K$-increasing and continuous sequence of models $\langle M_i \in K_\mu \mid i < \sigma\rangle$ such that

(1) $M = M_0$,
(2) $M' = \bigcup_{i<\sigma} M_i$
(3) for $i < \sigma$, M_i is an amalgamation base and
(4) M_{i+1} is universal over M_i.

Remark 2.24. (1) Notice that in Definition 2.23 for $i < \sigma$ and i a limit ordinal, M_i is a (μ, i)-limit model.
(2) Notice that Condition (4) implies Condition (3) of Definition 2.23.

Definition 2.25. We say that M' is a (μ,σ)-limit iff there is some $M \in K$ such that M' is a (μ,σ)-limit over M.

Notation 2.26. (1) For μ a cardinal and σ a limit ordinal with $\sigma < \mu^+$, we write K^σ_μ for the collection of (μ,σ)-limit models of K.
(2) We define

$$K^*_\mu := \{M \in K \mid M \text{ is a } (\mu,\sigma)\text{-limit model for some limit ordinal } \sigma < \mu^+\}$$

as the collection of limit models of K.

Limit models also exist in certain abstract elementary classes. By repeated applications of Lemma 2.21 the existence of (μ,ω)-limit models can be proved:

Proposition 2.27 (Theorem 1.3.1 from [ShVi]). Let μ be a cardinal such that $\mu < \lambda$. For every $M \in K^{am}_\mu$, there exists $M' \in K$ such that $M \prec_K M'$ and M' is a (μ,ω)-limit over M.

In order to extend this argument further to yield the existence of (μ,θ)-limits for arbitrary limit ordinals $\theta<\mu^+$, we need to be able to verify that limit models are in fact amalgamation bases. We will examine this in Section 4.

While the existence of certain limit models is relatively easy to derive from the categoricity assumption, the uniqueness of limit models is more difficult. Here we recall two easy uniqueness facts which state that limit models of the same length are isomorphic:

Proposition 2.28 (Fact 1.3.6 from [ShVi]). *Let $\mu\geq LS(K)$ and $\theta<\mu^+$. If M_1 and M_2 are (μ,θ)-limits over M, then there exists an isomorphism $g:M_1\rightarrow M_2$ such that $g\restriction M=id_M$. Moreover if M_1 is a (μ,θ)-limit over M_0, N_1 is a (μ,θ)-limit over N_0 and $g:M_0\cong N_0$, then there exists a $\prec_K$-mapping, $\hat{g}$, extending g such that $\hat{g}:M_1\cong N_1$.*

Proposition 2.29 (Fact 1.3.7 from [ShVi]). *Let μ be a cardinal and θ a limit ordinal with $\theta<\mu^+\leq\lambda$. If M is a (μ,θ)-limit model, then M is a $(\mu,cf(\theta))$-limit model.*

A more challenging uniqueness question, which was a major goal of [ShVi], is:

Conjecture 2.30. *Suppose K is categorical in some $\lambda>LS(K)$. Let $\mu\geq LS(K)$ and $\theta_1,\theta_2<\mu^+\leq\lambda$ be given. If M_1 is a (μ,θ_1)-limit over M and M_2 is a (μ,θ_2)-limit over M, then there exists $g:M_1\cong M_2$ with $g\restriction M=id_M$.*

A main result of this paper, Corollary 10.1, is a solution to this conjecture under Assumptions 1.2(1)-(5) plus K^{am}_μ is closed under unions of length $<\mu^+$.

We will need one more notion of limit model, which will appear implicitly in the proofs of Theorem 6.8 and Theorem 7.10. This notion is a mild extension of the notion of limit models already defined:

Definition 2.31. Let μ be a cardinal $<\lambda$, we say that $\check{M}$ is a (μ,μ^+)-limit over M if there exists a $\prec_K$-increasing and continuous chain of models $\langle M_i\in K^{am}_\mu\mid i<\mu^+\rangle$ satisfying

(1) $M_0=M$
(2) $\bigcup_{i<\mu^+}M_i=\check{M}$ and
(3) for $i<\mu^+$, M_{i+1} is universal over M_i

Remark 2.32. While it is known that (μ,θ)-limit models are amalgamation bases when $\theta<\mu^+$, it is open as to whether or not (μ,μ^+)-limits are amalgamation bases. To avoid confusion between these two concepts of limit models, we will always denote (μ,μ^+)-limit models with a $\check{}$ above the model's name (ie $\check{M}$).

The existence of (μ,μ^+)-limit models follows from the fact that (μ,θ)-limit models are amalgamation bases when $\theta<\mu^+$, see Corollary 4.9. The uniqueness of (μ,μ^+)-limit models (Proposition 2.33) can be shown using an easy back and forth construction as in the proof of Proposition 2.28.

Proposition 2.33. Suppose M^*_1 and M^*_2 are (μ,μ^+)-limits over M_1 and M_2, respectively. If there exists an isomorphism $h: M_1 \cong M_2$, then h can be extended to an isomorphism $g: M^*_1 \cong M^*_2$.

(μ,μ^+)-limit models turn out to be useful as replacements for monster models:

Proposition 2.34. If M^* is a (μ,μ^+)-limit, then for every $N \prec_K M^*$ with $N \in K^{am}_\mu$, we have that M^* is universal over N. Moreover, M^* is a (μ,μ^+)-limit over N.

3. Ehrenfeucht-Mostowski Models

Since K has no maximal models, K has models of cardinality $\geq$ Hanf(K). Then by Theorem 3.1, we can construct Ehrenfeucht-Mostowski models.

Theorem 3.1 (Claim 0.6 of [Sh394] or see [Gr]). Assume that K is an AEC that contains a model of cardinality $\beth_{(2^{LS(K)})^+}$. Then, there is a Φ, proper for linear orders, such that for linear orders $I \subseteq J$ we have that

(1) $EM(I,\Phi)\restriction L(K) \prec_K EM(J,\Phi)\restriction L(K)$ and
(2) $\|EM(I,\Phi)\restriction L(K)\| = |I| + LS(K)$.

We describe an index set which appears often in work toward the categoricity conjecture. This index set was used in [KoSh], [Sh394] and [ShVi].

Notation 3.2. Let $\alpha < \kappa$ be given. We define

$$I_\alpha := \left\{\eta \in {}^\omega\alpha : |\{n<\omega \mid \eta[n] \neq 0\}| \text{ is finite}\right\}$$

Associate with I_α the lexicographical ordering. If $X \subseteq \alpha$, we write $I_X := \{\eta \in {}^\omega X : |\{n<\omega \mid \eta[n] \neq 0\}| \text{ is finite}\}$.

The following proposition is proved in several papers e.g. [ShVi].

Proposition 3.3. If $M \prec_K EM(I_\kappa,\Phi)\restriction L(K)$ is a model of cardinality μ with $\mu^+ < \kappa$, then there exists a $\prec_K$-mapping $f: M \to EM(I_{\mu^+},\Phi)\restriction L(K)$.

A variant of this universality property is (implicit in Lemma 3.7 of [KoSh]):

Proposition 3.4. Suppose χ is a regular cardinal. If $M \prec_K EM(I_\chi,\Phi)\restriction L(K)$ is a model of cardinality χ and $N \prec_K EM(I_\kappa,\Phi)\restriction L(K)$ is an extension of M of cardinality $\|M\|$, then there exists a $\prec_K$-embedding $f: N \to EM(I_\chi,\Phi)\restriction L(K)$ such that $f\restriction M = id_M$.

4. Amalgamation Bases

Since the amalgamation property for abstract elementary classes is inherent in the definition of types, most work towards understanding AECs has been under the assumption that the class K has the amalgamation property. In [ShVi], Shelah and Villaveces begin to tackle the categoricity problem with an approach that does not require the amalgamation property as an assumption. Shelah and Villaveces, however, prove a weak amalgamation property, which they refer to as density of amalgamation bases, summarized here:

Theorem 4.1 (Theorem 1.2.4 from [ShVi]). For every $M \in K_{<\lambda}$, there exists $N \in K^{am}_{\|M\|}$ with $M \prec_K N$.

We can now improve Lemma 2.21 slightly. This improvement is used throughout this paper.

Lemma 4.2. For every μ with $LS(K) < \mu < \lambda$, if $M \in K^{am}_\mu$, $N \in K$ and $\bar a \in {}^{\mu^+>}N$ are such that $M \prec_K N$, then there exists $M^* \in K^{am}_\mu$ such that M^* is universal over M and $M \cup \bar a \subseteq M^*$.

Proof. By Axiom 6 of AEC, we can find $M^0 \prec_K N$ of cardinality μ containing $M \cup \bar a$. Applying Theorem 4.1, there exists an amalgamation base of cardinality μ, say M^{00}, extending M^0. By Lemma 2.21 we can find a universal extension of M^{00} of cardinality μ, say M^*.

Notice that M^* is also universal over M. Why? Suppose M' is an extension of M of cardinality μ. Since M is an amalgamation base we can amalgamate M^{00} and M' over M. WLOG we may assume that the amalgam, M'', is an extension of M^{00} of cardinality μ and $f' : M' \to M''$ with $f' \restriction M = id_M$.

$$\begin{tikzcd}
M' \arrow[r, "f'"] & M'' \\
M \arrow[u, "id"] \arrow[r, "id"] & M^{00} \arrow[u, "id"']
\end{tikzcd}$$

Now, since M^* is universal over M^{00}, there exists a $\prec_K$-mapping g such that $g : M'' \to M^*$ with $g \restriction M^{00} = id_{M^{00}}$. Notice that $g \circ f'$ gives us the desired mapping of M' into M^*. $\dashv$

While Theorem 4.1 asserts the existence of amalgamation bases, it is unknown (in this context) what characterizes amalgamation bases. Shelah and Villaveces have claimed that every limit model is an amalgamation base (Fact 1.3.10 of [ShVi]), using $S^{\mu^+}_{\mathrm{cf}(\mu)}$. Notice this is more than the asssumption of GCH that they make throughout their paper. We believe that $S^{\mu^+}_{\mathrm{cf}(\mu)}$ is not sufficient to carry out the argument that they suggest. A stronger set theoretic assumption (namely the weak form of diamond listed as Assumption 1.2(5)) is needed. We provide a proof that every (μ,θ)-limit model with $\theta < \mu^+$ is an amalgamation base under this additional assumption:

Theorem 4.3. *Under Assumption 1.2 (specifically under the set theoretic assumption of $\Phi_{\mu^+}(S^{\mu^+}_\theta)$ for every regular $\theta < \mu^+$), if M is a (μ,θ)-limit for some θ with $\theta < \mu^+ \leq \kappa$, then M is an amalgamation base.*

Let us first recall some set theoretic definitions and facts concerning the weak diamond.

Definition 4.4. Let θ be a regular cardinal $< \mu^+$. We denote

$$S^{\mu^+}_\theta := \{\alpha < \mu^+ \mid cf(\alpha) = \theta\}.$$

Definition 4.5. For μ a cardinal and $S \subseteq \mu^+$ a stationary set, $\Phi_{\mu^+}(S)$ is said to hold iff for all $F : {}^{\mu^+>}2 \to 2$ there exists $g : \mu^+ \to 2$ so that for every $f : \mu^+ \to 2$ the set

$$\{\delta \in S \mid F(f \restriction \delta) = g(\delta)\} \text{ is stationary}$$

We will be using a consequence of $\Phi_{\mu^+}(S)$, called $\Theta_{\mu^+}(S)$ (see [Gr]).

Definition 4.6. For μ a cardinal $\geq \aleph_1$ and $S \subseteq \mu^+$ a stationary set, $\Theta_{\mu^+}(S)$ is said to hold if and only if for all families of functions

$$\{f_\eta : \eta \in {}^{\mu^+}2 \text{ where } f_\eta : \mu^+ \to \mu^+\}$$

and for every club $C \subseteq \mu^+$, there exist $\eta \neq \nu \in {}^{\mu^+}2$ and there exists $\delta \in C \cap S$ such that

(1) $\eta \restriction \delta = \nu \restriction \delta$,
(2) $f_\eta \restriction \delta = f_\nu \restriction \delta$ and
(3) $\eta[\delta] \neq \nu[\delta]$.

The following implications (Fact 4.7) are a consequence of work of Devlin and Shelah [DS]. For an exposition of Fact 4.7 see [Gr].

Fact 4.7. $2^\mu < 2^{\mu^+} \Rightarrow \Phi_{\mu^+}(S^{\mu^+}_\theta) \Rightarrow \Theta_{\mu^+}(S^{\mu^+}_\theta)$.

Before we begin the proof of Theorem 4.3, notice that:

Remark 4.8 (Invariance)**.** By Axiom 1 of AEC, if M is an amalgamation base and f is an $\prec_K$-embedding, then $f(M)$ is an amalgamation base.

Proof of Theorem 4.3. Given μ, suppose that θ is the minimal infinite ordinal $< \mu^+$ such that there exists a model M which is a (μ,θ)-limit and not an amalgamation base. Notice that by Proposition 2.9, we may assume that $cf(\theta) = \theta$.

Now we define by induction on the length of $\eta \in {}^{\mu^+>}2$ a tree of structures, $\langle M_\eta \mid \eta \in {}^{\mu^+>}2\rangle$, satisfying:

(1) for $\eta \lhd \nu \in {}^{\mu^+>}2$, $M_\eta \prec_K M_\nu$

(2) for $l(\eta)$ a limit ordinal with $cf(l(\eta)) \leq \theta$, $M_\eta = \bigcup_{\alpha<l(\eta)} M_{\eta\upharpoonright\alpha}$

(3) for $\eta \in {}^{\alpha}2$ with $\alpha \in S^{\mu^+}_{\theta}$,

  (a) M_η is a (μ,θ)-limit model

  (b) $M_{\eta^\frown 0}, M_{\eta^\frown 1}$ cannot be amalgamated over M_η.

  (c) $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ are amalgamation bases of cardinality μ,

(4) for $\eta \in {}^{\alpha}2$ with $\alpha \notin S^{\mu^+}_{\theta}$,

  (a) M_η is an amalgamation base

  (b) $M_{\eta^\frown 0}, M_{\eta^\frown 1}$ are universal over M_η and

  (c) $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ are amalgamation bases of cardinality μ (it may be that $M_{\eta^\frown 0} = M_{\eta^\frown 1}$ in this case).

This construction is possible:

$\eta = \langle\rangle$: By Theorem 4.1, we can find $M^0 \in K^{am}_\mu$ such that $M \prec_K M^0$. Define $M_{\langle\rangle} := M^0$.

$l(\eta)$ is a limit ordinal: When $cf(l(\eta)) > \theta$, let $M'_\eta := \bigcup_{\alpha<l(\eta)} M_{\eta\upharpoonright\alpha}$. M'_η is not necessarily an amalgamation base, but for the purposes of this construction, continuity at such limits is not important. Thus we can find an extension of M'_η, say M_η, of cardinality μ where M_η is an amalgamation base.

For η with $cf(l(\eta)) \leq \theta$, we require continuity. Define $M_\eta := \bigcup_{\alpha<l(\eta)} M_{\eta\upharpoonright\alpha}$. We need to verify that if $l(\eta) \notin S^{\mu^+}_{\theta}$, then M_η is an amalgamation base. In fact, we will show that such a M_η will be a $(\mu, cf(l(\eta)))$-limit model. Let $\langle \alpha_i \mid i < cf(l(\eta))\rangle$ be an increasing and continuous sequence of ordinals converging to $l(\eta)$ such that $cf(\alpha_i) < \theta$ for every $i < cf(l(\eta))$. Condition (4b) guarantees that for $i < cf(l(\eta))$, $M_{\eta\upharpoonright\alpha_{i+1}}$ is universal over $M_{\eta\upharpoonright\alpha_i}$. Additionally, condition (2) ensures that $\langle M_{\eta\upharpoonright\alpha_i} \mid i < cf(l(\eta))\rangle$ is continuous. This sequence of models witnesses that M_η is a $(\mu, cf(l(\eta)))$-limit model. By our minimal choice of θ, we have that $(\mu, cf(l(\eta)))$-limit models are amalgamation bases.

$\eta^\frown i$ where $l(\eta) \in S^{\mu^+}_{\theta}$: We first notice that $M_\eta := \bigcup_{\alpha<l(\eta)} M_{\eta\upharpoonright\alpha}$ is a (μ,θ)-limit model. Why? Since $l(\eta) \in S^{\mu^+}_{\theta}$ and θ is regular, we can find an increasing and continuous sequence of ordinals $\langle \alpha_i \mid i < \theta\rangle$ converging to $l(\eta)$ such that for each $i < \theta$ we have that $cf(\alpha_i) < \theta$. Condition (4b) of the construction guarantees that for each $i < \theta$, $M_{\eta\upharpoonright\alpha_{i+1}}$ is universal over $M_{\eta\upharpoonright\alpha_i}$. Thus $\langle M_{\eta\upharpoonright\alpha_i} \mid i < \theta\rangle$ witnesses that M_η is a (μ,θ)-limit model.

Since M_η is a (μ,θ)-limit, we can fix an isomorphism $f : M \cong M_\eta$. By Remark 4.8, M_η is not an amalgamation base. Thus there exist $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ extensions of M_η which cannot be amalgamated over M_η. WLOG we can choose $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ to be elements of K^{am}_μ.

$\eta^\frown i$ where $l(\eta) \notin S^{\mu^+}_{\theta}$: Since M_η is an amalgamation base, we can choose $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ to be extensions of M_η such that $M_{\eta^\frown l} \in K^{am}_\mu$ and $M_{\eta^\frown l}$ is universal over M_η, for $l = 0, 1$.

This completes the construction. For every $\eta \in {}^{\mu^+}2$, define $M_\eta := \bigcup_{\alpha<\mu^+} M_{\eta\restriction\alpha}$. By categoricity in μ, and Proposition 3.3 we can fix a $\prec_K$-mapping $g_\eta : M_\eta \to EM(I_{\mu^+},\Phi)\restriction L(K)$ for each $\eta \in {}^{\mu^+}2$. Now apply $\diamondsuit_{\mu^+}(S^{\mu^+})$ to find $\eta,\nu \in {}^{\mu^+}2$ and $\alpha \in S^{\mu^+}$ such that

- $\zeta := \eta\restriction\alpha = \nu\restriction\alpha$,
- $\eta[\alpha] = 0$, $\nu[\alpha] = 1$ and
- $g_\eta\restriction M_\zeta = g_\nu\restriction M_\zeta$.

By Axiom 6 (the Löwenheim-Skolem property) of AEC, there exists $N \prec_K EM(I_{\mu^+},\Phi)\restriction L(K)$ of cardinality μ such that the following diagram commutes:

$$\begin{array}{ccc}
M_{\zeta^\frown 1} & \xrightarrow{\;g_\nu\restriction M_{\zeta^\frown 1}\;} & N \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle g_\eta\restriction M_{\zeta^\frown 0}} \\
M_\zeta & \xrightarrow[\mathrm{id}]{} & M_{\zeta^\frown 0}
\end{array}$$

Notice that $g_\eta\restriction M_{\zeta^\frown 0}$, $g_\nu\restriction M_{\zeta^\frown 1}$ and N witness that $M_{\zeta^\frown 0}$ and $M_{\zeta^\frown 1}$ can be amalgamated over M_ζ. Since $l(\zeta) = \alpha \in S^{\mu^+}$, we contradict condition (3b) of the construction.

⊣

Corollary 4.9 (Existence of limit models and (μ,μ^+)-limit models). For every cardinal μ, and limit ordinal θ with $\theta \leq \mu^+$, if M is an amalgamation base of cardinality μ, then there exists $M' \in K^{am}_\mu$ which is a (μ,θ)-limit over M.

Proof. By repeated applications of Lemma 2.21 and Theorem 4.3. ⊣

5. Weak Disjoint Amalgamation

Shelah and Villaveces prove a version of weak disjoint amalgamation in an attempt to prove an extension property for towers. We will be using weak disjoint amalgamation and provide a proof here for completeness.

Theorem 5.1 (Weak Disjoint Amalgamation [ShVi]). Given $\mu \geq LS(K)$ and $\alpha, \theta_0 < \mu^+$ with θ_0 regular. If M_0 is a (μ,θ_0)-limit and $M_1, M_2 \in K_\mu$ are $\prec_K$-extensions of M_0, then for every $\bar b \in {}^\alpha(M_1 \setminus M_0)$, there exist M_3, a model, and h, a $\prec_K$-embedding, such that

(1) $h : M_2 \to M_3$;
(2) $h\restriction M_0 = \mathrm{id}_{M_0}$ and
(3) $h(M_2) \cap \bar b = \emptyset$ (equivalently $h(M_2) \cap M_1 = M_0$).

Shelah and Villaveces provide a proof of this theorem in [ShVi]. It has been suggested that I elaborate on the proof here. John Baldwin may have a simplification of this proof.

Proof. Suppose that M_0, M_1, M_2 and $b \in M_1$ form a counter-example. Since M_0 is a μ amalgamation base, we may assume that there exists $M^* \in K_\mu$ with $M_1, M_2 \prec_K M^*$. Let θ be regular and $\theta < \mu^+$ such that M_0 is a (μ, θ)-limit. We define a $\prec_K$-increasing and continuous sequence of models $\langle N_i \mid i < \mu^+ \rangle$ satisfying:

(1) $N_i \in K^{am}_\mu$
(2) N_{i+1} is universal over N_i and
(3) when $cf(i) = \theta$, we additionally define N^1_i, N^2_i, N^*_i and $b_i \in N^1_i$ such that there exists an isomorphism $f_i : M^* \cong N^*_i$ with $f_i(M_0) = N_i$, $f_i(M_1) = N^1_i$, $f_i(M_2) = N^2_i$ and $f_i(b) = b_i$.

The construction is possible by Lemma 2.21, Theorem 4.3 and Proposition 2.28.

Let $N_{\mu^+} := \bigcup_{i<\mu^+} N_i$. Since K is categorical in λ, Proposition 3.3 allows us to find a $\prec_K$-mapping $g : N_{\mu^+} \to EM(I^+_\mu; \Phi)_\mu \upharpoonright L(K)$. So WLOG, we may assume that $N_{\mu^+} \prec_K EM(I^+_\mu; \Phi)_\mu \upharpoonright L(K)$.

Let $E \subseteq \mu^+$ be a club such that

$$\delta \in E \Rightarrow N_\delta \prec_K EM(I_\delta; \Phi)_\mu \upharpoonright L(K):$$

For each $i \in S^{\mu^+}_\theta$, choose a Skolem-term τ_i and a sequence of indices $\alpha_{i,0}, \dots, \alpha_{i,n_i-1}$ such that $b_i = \tau_i(\alpha_{i,0}, \dots, \alpha_{i,n_i-1})$. Let $m_i < n_i$ be such

$$k < m_i \iff \alpha_{i,k} \in I_i:$$

Set $\alpha_{i,<m_i} := \langle \alpha_{i,k} \mid 0 \leq k < m_i \rangle$ and $\alpha_{i,\geq m_i} := \langle \alpha_{i,k} \mid m_i \leq k < n_i \rangle$.

Let $\gamma_0 \in E \cap S^{\mu^+}_\theta$.

For every γ_1, with $\gamma_0 < \gamma_1 < \mu^+$. Define g_{γ_1} to be the $\prec_K$-mapping from $EM(I_{\gamma_1}; \Phi)_\mu \upharpoonright L(K)$ to $EM(I_{\mu^+}; \Phi)_\mu \upharpoonright L(K)$ induced by the mapping from μ^+ to μ^+ defined by

$$j \mapsto \begin{cases} j & \text{if } j < \gamma_0 \\ \gamma_1 + j & \text{if } \gamma_0 \leq j < \gamma_1 \end{cases}$$

Subclaim 5.2. There exists some $\gamma_1 < \mu^+$ such that $g_{\gamma_1}(N^1_{\gamma_0}) \cap b_{\gamma_0} = \emptyset$.

Proof. Suppose the claim fails. Then for every δ with $\gamma_0 < \delta < \mu^+$, there exists a Skolem term τ_δ and a sequence of elements of I_δ

$$\beta_{\delta,0}, \dots, \beta_{\delta,m_\delta-1}, \beta_{\delta,m_\delta}, \dots, \beta_{\delta,n_\delta-1}$$

such that

$$k < m_\delta \iff \beta_{\delta,k} \in I_{\gamma_0}$$

and $b = \tau_\delta(\beta_{\delta,0}, \dots, \beta_{\delta,n_\delta-1})$.

Let $\beta_{\delta,<m_\delta} := \langle \beta_{\delta,k} \mid 0 \leq k < m_\delta \rangle$ and $\beta_{\delta,\geq m_\delta} := \langle \beta_{\delta,k} \mid m_\delta \leq k < n_\delta \rangle$.

Notice that

$$EM(I_{\mu^+}; \Phi)_\mu \upharpoonright L(K) \models b = \tau_{\gamma_0}(\beta_{\delta,<m_\delta}, \beta_{\delta,\geq m_\delta}) = \tau_{\gamma_0}(\alpha_{\gamma_0,<m_{\gamma_0}}, \alpha_{\gamma_0,\geq m_{\gamma_0}}):$$

By our definition of g_δ, we have that

$$(*)_\delta \quad k \geq m_\delta \iff \beta_{\delta,k} \in I_{\delta \setminus \gamma_1 \cup \gamma_0}:$$

In other words when $k \geq m_\delta$, every term from the sequence $\beta_{\delta,k}$ which is larger than δ_0 is also larger than δ_1. Thus, for $k \geq m_\delta$, the ordinals in $\beta_{\delta,k}$ above δ_0 are all greater than the ordinals above δ_0 appearing in the sequence $\alpha_{\delta_0,\geq m_{\delta_0}}$.

Since all our indices are finite sequences and δ_0 is a limit ordinal, there exists $\epsilon < \delta_1$ such that $\alpha_{\delta_0,<m_{\delta_0}}, \beta_{\delta,<m_\delta} \in I_\epsilon$. This allows us to find a sequence $\alpha^\epsilon \in I_{\delta_0}$ which has the same type over I_ϵ (with respect to the lexicographical ordering) as $\alpha_{\delta_0,\geq m_{\delta_0}}$. So by indiscernibility and $(*)_\delta$

$$EM(I_{\mu^+},\Phi)\restriction L(K) \models \sigma_{\delta_0}(\beta_{\delta,<m_\delta},\beta_{\delta,\geq m_\delta}) = \tau_{\delta_0}(\alpha_{\delta_0,<m_{\delta_0}},\alpha^\epsilon).$$

Notice that $\sigma_{\delta_0}(\beta_{\delta,<m_\delta},\beta_{\delta,\geq m_\delta}) = b$. Thus we have found a way to construct b from I_{δ_0} (by $\tau_{\delta_0}(\alpha_{\delta_0,<m_{\delta_0}},\alpha^\epsilon)$). This contradicts our choice of $b \notin EM(I_{\delta_0})\restriction L(K)$.

$\dashv$

Let δ_1 be as in Subclaim 5.2. There exists an ordinal $\alpha_2 < \mu^+$ such that $g_{\delta_1} : \delta_1 \to \alpha_2$. Let g be the $\prec_K$-mapping induced by g_{δ_1} such that $g : N_{\delta_1} \to EM(I_{\alpha_2},\Phi)\restriction L(K)$. Notice that by our choice of δ_1, we have that g and $EM(I_{\alpha_2},\Phi)\restriction L(K)$ witnesses that $N^1_{\delta_0}, N^2_{\delta_0}, N^3_{\delta_0}$ and b_{δ_0} can be weakly disjointly amalgamated.

$\dashv$

Let us state an easy corollary of Theorem 5.1 that will simplify future constructions:

Corollary 5.3. Suppose μ, M_0, M_1, M_2 and b are as in the statement of Theorem 5.1. If M^* is universal over M_1, then there exists a $\prec_K$-mapping h such that

(1) $h : M_2 \to M^*$,
(2) $h\restriction M_0 = id_{M_0}$ and
(3) $h(M_2) \cap b = \emptyset$ (equivalently $h(M_2) \cap M_1 = \emptyset$).

Proof. By Theorem 5.1, there exists a $\prec_K$-mapping g and a model M_3 of cardinality μ, such that

- $g : M_2 \to M_3$
- $g\restriction M_0 = id_{M_0}$
- $g(M_2) \cap b = \emptyset$ and
- $M_1 \prec_K M_3$.

Since M^* is universal over M_1, we can fix a $\prec_K$-mapping f such that

- $f : M_3 \to M^*$ and
- $f\restriction M_1 = id_{M_1}$

Notice that $h := g \circ f$ is the desired mapping from M_2 into M^*.

$\dashv$

6. $<^b_{\mu,\alpha}$-extension property for $K^*_{\mu,\alpha}$

Shelah introduced towers in [Sh48] and [Sh87b] as a tool to build a model of cardinality λ^+ from models of cardinality λ. Here we will use the towers to prove the uniqueness of limit models by producing a model which is simultaneously a (μ,θ_1)-limit model and a (μ,θ_2)-limit model. The construction of such a model is sufficient to prove the uniqueness of limit models by Proposition 2.28.

Definition 6.1 (Towers Definition 3.1.1 of [ShVi]). Let $\mu > LS(K)$ and $\alpha < \mu^+$.

(1)
$$K_{\mu,\alpha} := \left\{ (\bar M,\bar a) \;\middle|\; \begin{array}{l} (\bar M,\bar a) := (\langle M_i \mid i<\alpha\rangle, \langle a_i \mid i<\alpha\rangle); \\ \bar M \text{ is } \prec_K\text{-increasing}; \\ \text{for every } i<\alpha,\ a_i \in M_{i+1}\setminus M_i; \\ \text{for every } i<\alpha,\ M_i \in K_\mu \end{array} \right\}$$

(2) $K^\theta_{\mu,\alpha} := \{(\bar M,\bar a)\in K_{\mu,\alpha} \mid \text{for every } i<\alpha,\ M_i \text{ is a } (\mu,\theta)\text{-limit}\}$

(3) $K^*_{\mu,\alpha} := \bigcup_{\theta<\mu^+} K^\theta_{\mu,\alpha}$

Fact 6.2 (Fact 3.1.7 from [ShVi]). Suppose K is categorical in λ. Given $\lambda > \mu \geq LS(K)$, $\alpha < \mu^+$ and θ a regular cardinal with $\theta < \mu^+$, we have that $K^\theta_{\mu,\alpha} \neq \emptyset$.

Roughly speaking, we will construct an array of models of width θ_1 and height θ_2 in such a way that the union will simultaneously be a (μ,θ_1)-limit model and a (μ,θ_2)-limit model. Each row in our array will be a tower from K^*_{μ,θ_1}. We define the array by induction on the height (θ_2) by finding an "increasing" and continuous chain of towers from K^*_{μ,θ_1}. We need to make explicit what we mean by "increasing". One property that the ordering on towers should have is that the union of an "increasing" chain of towers from K^*_{μ,θ_1} should also be a member of K^*_{μ,θ_1}. In particular we need to guarantee that the models that appear in the union be limit models. This motivates the following ordering on towers:

Definition 6.3 (Definition 3.1.3 of [ShVi]). For $(\bar M,\bar a),(\bar N,\bar b) \in K^*_{\mu,\alpha}$ we say that

(1) $(\bar M,\bar a) \leq^b_{\mu,\alpha} (\bar N,\bar b)$ if and only if
   (a) $\bar a = \bar b$;
   (b) for every $i<\alpha$, $M_i \preceq_K N_i$ and
   (c) whenever $M_i \prec_K N_i$, then N_i is universal over M_i.

(2) $(\bar M,\bar a) <^b_{\mu,\alpha} (\bar N,\bar b)$ if and only if $(\bar M,\bar a) \leq^b_{\mu,\alpha} (\bar N,\bar b)$ and for every $i<\alpha$, $M_i \neq N_i$.

Remark 6.4. If $\langle(\bar M,\bar a)^\gamma \in K^*_{\mu,\alpha} \mid \gamma<\delta\rangle$ is a $<^b_{\mu,\alpha}$-increasing and continuous chain with $\delta < \mu^+$, then $\bigcup_{\gamma<\delta}(\bar M,\bar a)^\gamma \in K^*_{\mu,\alpha}$. Why? Notice that for $i<\alpha$, $M^\delta_i := \bigcup_{\gamma<\delta} M^\gamma_i$ is a limit model witnessed by $\langle M^\gamma_i \mid \gamma<\delta\rangle$.

In order to construct a non-trivial chain of towers, we need to be able to take proper $<^b_{\mu,\alpha}$-extensions.

Definition 6.5. We say the $<^b_{\mu,\alpha}$-extension property holds if for every $(\bar M,\bar a,\bar N)\in K^*_{\mu,\alpha}$ there exists $(\bar M',\bar a,\bar N)\in K^*_{\mu,\alpha}$ such that $(\bar M,\bar a,\bar N)<^b_{\mu,\alpha}(\bar M',\bar a,\bar N)$.

Remark 6.6. Shelah and Villaveces claim the $<^b_{\mu,\alpha}$-extension property as Fact 3.19(1) in [ShVi]. Their proof does not converge. As of the Fall of 2001, they were unable to produce a proof of this claim.

We will prove the $<^b_{\mu,\alpha}$-extension property for a particular class of towers:

Definition 6.7. $(\langle M_i\mid i<\alpha\rangle,\bar a,\bar N)\in K^*_{\mu,\alpha}$ is nice provided that for every limit ordinal $i<\alpha$, we have that $\bigcup_{j<i}M_j$ is an amalgamation base.

Notice that in the definition of towers, we do not require continuity at limit ordinals of the sequence of models. This allows for towers in which $M_i\neq\bigcup_{j<i}M_j$. Since we only require that M_i is an amalgamation base, there are towers which are not necessarily nice.

Theorem 6.8 (The $<^b_{\mu,\alpha}$-extension property for nice towers)**.** For every nice $(\bar M,\bar a,\bar N)\in K^*_{\mu,\alpha}$, there exists a nice tower $(\bar M',\bar a,\bar N)\in K^*_{\mu,\alpha}$ such that $(\bar M,\bar a,\bar N)<^b_{\mu,\alpha}(\bar M',\bar a,\bar N)$

Remark 6.9. Why isn't Theorem 6.8 sufficient? What is needed to conclude the uniqueness of limit models?

1. (1) We would like to ultimately construct a $<^b_{\mu,\alpha}$-increasing and continuous chain of towers. The problem occurs at limit stages. Suppose we have constructed a $<^b_{\mu,\alpha}$-increasing and continuous sequence of towers $\langle(\bar M^n,\bar a,\bar N)\mid n\leq\omega\rangle$. We may not be able to extend this chain of towers because we are not guaranteed that the limit $(\bar M^\omega,\bar a,\bar N)$ is a nice tower. A sufficient resolution would be to show that every tower has a $\leq^b_{\mu,\alpha}$-extension which is nice.
2. (2) Alternatively, if we try to carry out the arguments of [ShVi] towards the uniqueness of limit models using only nice towers, we see that the only obstacle is to prove that reduced towers are continuous.
3. (3) If we make an additional assumption that K^{am}_μ is closed under unions of length $<\mu^+$, then the full $<^b_{\mu,\alpha}$-extension property follows from Theorem 6.8 since every tower in $K^*_{\mu,\alpha}$ would be nice.

Proof of Theorem 6.8. Let μ be a cardinal and α a limit ordinal such that $\alpha<\mu^+\leq\kappa$. Let a nice tower $(\bar M,\bar a,\bar N)\in K^*_{\mu,\alpha}$ be given. Denote by M_α a model in K^{am}_μ extending $\bigcup_{i<\alpha}M_i$. Fix M^* to be a (μ,μ^+)-limit model over M_α.

We define by induction on $i<\alpha$ a sequence of models $\langle M^0_i\mid i<\alpha\rangle$ and a sequence of $\prec_K$-mappings $\langle f^0_{j,i}\mid j<i<\alpha\rangle$ and $\langle f^*_{j,i}\mid j<i<\alpha\rangle$ such that for $i\leq\alpha$:

(1) $(\langle f^0_{j,i}(M^0_j)\mid j\leq i\rangle,\bar a\restriction i,\bar N\restriction i)$ is a $<^b_{\mu,i}$-extension of $(\bar M,\bar a,\bar N)\restriction i$,

(2) $\langle M'_j \mid j < i\rangle, \langle f'_{j,i} \mid j \leq i\rangle$ forms a directed system,
(3) M'_i is universal over M_i,
(4) M'_{i+1} is universal over $f'_{i,i+1}(M'_i)$,
(5) $f'_{j,i}\restriction M_j = id_{M_j}$,
(6) $M'_i \prec_K M^*$,
(7) $f'_{j,i}$ can be extended to an automorphism of M^*, $f^*_{j,i}$, for $j \leq i$ and
(8) $\langle M^*_j = M^* \mid j < i\rangle, \langle f^*_{k,j} \mid k \leq j < i\rangle$ forms a directed system.

Notice that the M'_i's will not necessarily form an extension of the tower $(\bar M, \bar a)$. Rather, for each $i < \alpha$, we find some image of $\langle M_j \mid j < i\rangle$ which will extend the initial segment of length i of $(\bar M, \bar a)$ (see condition (1) of the construction).

The construction is possible:

$i = 0$: Since M_0 is an amalgamation base, we can find $M''_0 \in K^{am}_\mu$ (a first approximation of the desired M'_0) such that M''_0 is universal over M_0. By Corollary 5.3 (applied to M_0, M_α, M''_0 and $\bar a$), we can find a $\prec_K$-mapping $h : M''_0 \to M^*$ such that $h\restriction M_0 = id_{M_0}$ and $h(M''_0) \cap \bar a = \emptyset$. Set $M'_0 := h(M''_0)$, $f'_{0,0} := id_{M'_0}$ and $f^*_{0,0} := id_{M^*}$.

$i = j+1$: Suppose that we have completed the construction for all $k \leq j$. Since M'_j and M_{j+1} are both $\prec_K$-substructures of M^*, we can get M''_{j+1} (a first approximation to the desired M'_{j+1}) such that $M''_{j+1} \in K^{am}_\mu$ is universal over M'_j and universal over M_{j+1}. How? By the Downward Löwenheim Skolem Axiom (Axiom 6) of AEC and the density of amalgamation bases (Theorem 4.1), we can find an amalgamation base L of cardinality μ such that $M'_j, M_{j+1} \prec_K L$. By Lemma 2.21 and Corollary 4.9 there exists M''_{j+1}, a (μ,ω)-limit over L.

Subclaim 6.10. M''_{j+1} is universal over M'_j and is universal over M_{j+1}.

Proof. It suffices to show that when $L_0 \prec_K L_1 \prec_K L$ are amalgamation bases of cardinality μ, if L is universal over L_1, then L is universal over L_0. Let L' be an extension of L_0 of cardinality μ. Since L_0 is an amalgamation base, we can find an amalgam L'' such that the following diagram commutes:

$$\begin{tikzcd} L' \arrow[r, "h"] & L'' \\ L_0 \arrow[u, "id"] \arrow[r, "id"] & L_1 \arrow[u, "id"'] \end{tikzcd}$$

Since L is universal over L_1, there exists $g : L'' \to L$ with $g\restriction L_1 = id_{L_1}$. Notice that $g \circ h : L' \to L$ with $g \circ h \restriction L_0 = id_{L_0}$. ⊣

M''_{j+1} may serve us well if it does not contain any a_l for $j+1 \leq l < \alpha$, but this is not guaranteed. So we need to make an adjustment. Notice that M^* is universal over M_{j+1}. Thus we can apply Corollary 5.3 to M_{j+1}, M_α, M''_{j+1} and $\langle a_l \mid j+1 \leq l < \alpha\rangle$. This yields a $\prec_K$-mapping h such that

• $h : M''_{j+1} \to M^*$

- $h \restriction M_{j+1} = \mathrm{id}_{M_{j+1}}$ and
- $h(M''_{j+1}) \cap \{a_l \mid j+1 \le l < \beta\} = \emptyset$.

Set $M'_{j+1} := h(M''_{j+1})$, $f'_{j+1,j+1} = \mathrm{id}_{M_{j+1}}$, $f^{\bullet}_{j+1,j+1} = \mathrm{id}_{M^{\bullet}}$ and $f'_{j,j+1} := h \restriction M'_j$. Since $M^{\bullet}$ is a (μ,μ^+)-limit over both M'_j and $f'_{j,j+1}(M'_j)$, by Proposition 2.33 we can extend $f'_{j,j+1}$ to an automorphism of $M^{\bullet}$, denoted by $f^{\bullet}_{j,j+1}$.

To guarantee that we have a directed system, for $k < j$, define $f'_{k,j+1} := f'_{j,j+1} \circ f'_{k,j}$ and $f^{\bullet}_{k,j+1} := f^{\bullet}_{j,j+1} \circ f^{\bullet}_{k,j}$.

i is a limit ordinal. Suppose that $(\langle M'_j \mid j < i\rangle, \langle f'_{k,j} \mid k \le j < i\rangle)$ and $(\langle M^{\bullet} \mid j < i\rangle, \langle f^{\bullet}_{k,j} \mid k \le j < i\rangle)$ have been defined. Since they are both directed systems, we can take direct limits.

Claim 6.11. We can choose the direct limits $(M^{*}_i, \langle f^{*}_{j,i} \mid j \le i\rangle)$ and $(M^{\bullet *}_i, \langle f^{\bullet *}_{j,i} \mid j \le i\rangle)$ of $(\langle M'_j \mid j < i\rangle, \langle f'_{k,j} \mid k \le j < i\rangle)$ and $(\langle M^{\bullet} \mid j < i\rangle, \langle f^{\bullet}_{k,j} \mid k \le j < i\rangle)$ respectively such that

- $M^{*}_i \prec_K M^{\bullet *}_i$
- $f^{\bullet *}_{j,i}$ is an automorphism of $M^{\bullet *}_i$ for every $j \le i$
- $M^{\bullet *}_i = M^{\bullet}$
- $f^{*}_{j,i} \restriction M_j = \mathrm{id}_{M_j}$ for every $j < i$.

Proof. We will realize each of these conditions one at a time. By Lemma 2.6 we may choose direct limits $(M^{**}_i, \langle f^{**}_{j,i} \mid j \le i\rangle)$ and $(M^{\bullet **}_i, \langle f^{\bullet **}_{j,i} \mid j \le i\rangle)$ such that $M^{**}_i \prec_K M^{\bullet **}_i$. By Claim 2.8 we have that for every $j \le i$, $f^{\bullet **}_{j,i}$ is an automorphism and $M^{\bullet **}_i = M^{\bullet}$. Notice that $(M^{**}_i, \langle f^{**}_{j,i} \mid j \le i\rangle)$ and $(M^{\bullet **}_i, \langle f^{\bullet **}_{j,i} \mid j \le i\rangle)$ form a direct limit satisfying the first three properties.

Subclaim 6.12. $\langle f^{**}_{j,i} \restriction M_j \mid j < i\rangle$ is increasing.

Proof. Let $j < k < i$ be given. By construction

$$f'_{j,k} \restriction M_j = \mathrm{id}_{M_j}.$$

An application of $f^{**}_{k,i}$ yields

$$f^{**}_{k,i} \circ f'_{j,k} \restriction M_j = f^{**}_{k,i} \restriction M_j.$$

By the definition of directed limits, we have

$$f^{**}_{j,i} \restriction M_j = f^{**}_{k,i} \circ f'_{j,k} \restriction M_j = f^{**}_{k,i} \restriction M_j.$$

$\dashv$

By the subclaim, we have that $g := \bigcup_{j<i} f^{**}_{j,i} \restriction M_j$ is a partial automorphism of $M^{\bullet}$ from $\bigcup_{j<i} M_j$ onto $\bigcup_{j<i} f^{**}_{j,i}(M_j)$. Since $M^{\bullet}$ is a (μ,μ^+)-limit model and since $\bigcup_{j<i} M_j$ is an amalgamation base we can extend g to $G \in \mathrm{Aut}(M^{\bullet})$ by Proposition 2.33. Notice this is the point of the proof where we use the assumption of niceness when we observe that $\bigcup_{j<i} M_j$ is an amalgamation base.

Now consider the direct limit defined by $M^\theta_i := G^{-1}(M^{\theta\theta}_i)$ with $\langle f^\theta_{j,i} := G^{-1}\circ f^{\theta\theta}_{j,i} \mid j < i\rangle$ and $f^\theta_{i,i} = id_{M^\theta_i}$ and the direct limit $M^{\bullet\theta}_i := M^\bullet$ with $\langle f^{\bullet\theta}_{j,i} := G^{-1}\circ f^{\bullet\theta\theta}_{j,i} \mid j < i\rangle$ and $f^{\bullet\theta}_{i,i} := id_{N^\theta_i}$. Notice that $f^\theta_{j,i}\restriction M_j = G^{-1}\circ f^{\theta\theta}_{j,i}\restriction M_j = id_{M_j}$ for $j < i$.

⊣

By Condition (4) of the construction, notice that M^θ_i is a (μ, i)-limit model witnessed by $\langle f^\theta_{j,i}(M'_j) \mid j < i\rangle$. Hence M^θ_i is an amalgamation base. Since M^θ_i and M_i both lie inside of $M^\bullet$, we can find $M''_i \in K^\theta_\mu$ which is universal over M_i and universal over M^θ_i.

By Corollary 5.3 applied to M_i, M_α, M''_i and $\langle a_l \mid l \leq i < \alpha\rangle$ we can find $h : M''_i \to M^\bullet$ such that $h\restriction M_i = id_{M_i}$ and $h(M''_i) \cap \{a_l \mid i \leq l < \alpha\} = \emptyset$.

Set $M'_i := h(M''_i)$, $f'_{i,i} := id_{M_{i,i}}$, $f^\bullet_{i,i} := id_{M^\bullet}$ and for $j < i$, $f'_{j,i} := h\circ f^\theta_{j,i}$. We need to verify that for $j \leq i$, $f'_{j,i}(M'_j) \cap \{a_l \mid j \leq l < \alpha\} = \emptyset$. Clearly by our application of weak disjoint amalgamation, we have that for every l with $i \leq l < \alpha$ and every $j \leq i$, $a_l \notin M'_i \supseteq f'_{j,i}(M'_j)$. Suppose that $j < i$ and l is such that $j \leq l < i$. By construction $a_l \notin f'_{j,l+1}(M'_j)$ and $f'_{l+1,i}(a_l) = a_l$. So $f'_{j,i}(M'_j) = f'_{l+1,i}\circ f'_{j,l+1}(M'_j)$ implies that $a_l \notin f'_{j,i}(M'_j)$.

Notice that for every $j < i$, $M^\bullet$ is a (μ, μ^+)-limit over both M'_j and $f'_{j,i}(M'_j)$. Thus by the uniqueness of (μ, μ^+)-limit models, we can extend $f'_{j,i}$ to an automorphism of $M^\bullet$, denoted by $f^\bullet_{j,i}$.

The construction is enough: Let M'_α and $\langle f_{i,\alpha} \mid i \leq \alpha\rangle$ be the direct limit of $(\langle M'_i \mid i < \alpha\rangle, \langle f_{j,i} \mid j \leq i < \alpha\rangle)$. By Subclaim 6.12 we may assume that $\bigcup_{i<\alpha} M_i \preceq_K M'_\alpha$. It is routine to verify that $(\langle f_{i,\alpha}(M'_i) \mid i < \alpha\rangle, a_\mu)$ is a $<^b_{\mu,\alpha}$-extension of (M^μ, a_μ).

⊣

7. $<^c_{\mu,\alpha}$ Extension Property for $K^\theta_{\mu,\alpha}$

Unfortunately, it seems that working with the relatively simple $K^\theta_{\mu,\alpha}$ towers is not sufficient to carry out the proof for the uniqueness of limit models. Shelah and Villaveces have identified a more elabrate tower. The extension proprety for these towers is also missing from [ShVi]. We provide a partial solution to this extension property, analagous to the solution for $K^\theta_{\mu,\alpha}$ in the previous section. In fact, we will have to further adjust our definition of towers to scattered towers in the following section. We introduce the scaled down towers of Sections 6 and 7 to break down the proof of the desired extension property into more manageable constructions.

We augment our towers with a dependence relation. The following variant of the first-order notion of splitting is often used in AECs. Most results relying on this notion are proved under the assumption of categoricity. Just recently progress has been made by considering μ-splitting in Galois-stable AECs in [GrVa].

Definition 7.1. Let μ be a cardinal with $\mu < \kappa$. For $M \in K^{am}$ and $p \in \text{ga-S}(M)$, we say that p μ-splits over N iff $N \prec_K M$ and there exist $N_1, N_2 \in K_\mu$ and a $\prec_K$-mapping $h : N_1 \cong N_2$ such that

(1) $h(p \restriction N_1) \neq p \restriction N_2$,
(2) $N \prec_K N_1, N_2 \prec_K M$ and
(3) $h \restriction N = id_N$.

Shelah and Villaveces draw a connection between categoricity and superstability-like properties by showing that under the assumption of categoricity there are no long splitting chains:

Theorem 7.2 (Theorem 2.2.1 from [ShVi]). Under Assumption 1.2 (1) through 1.2 (4), suppose that

(1) $\langle M_i \mid i \leq \delta \rangle$ is $\prec_K$-increasing and continuous,
(2) for all $i \leq \delta$, $M_i \in K^{am}_\mu$,
(3) for all $i < \delta$, M_{i+1} is universal over M_i
(4) $\mathrm{cf}(\delta) = \delta \leq \mu^+ \leq \lambda$, and
(5) $p \in \text{ga-S}(M_\delta)$.

Then there exists $i < \delta$ such that p does not μ-split over M_i.

Remark 7.3. The proof of Theorem 7.2 utilizes the full power of the categoricity assumption. In particular, Shelah and Villaveces use the fact that every model can be embedded into a reduct of an Ehrenfeucht-Mostowski model. It is open as to whether or not a similar theorem can be proven under the assumption of Galois-stability in every cardinality (Galois-superstability).

We derive the extension property for non-splitting types (Theorem 7.4). This result does not rely on the categoricity assumption. It is useful for developing stability theory for tame abstract elementary classes (see [GrVa]).

Theorem 7.4 (Extension of non-splitting types). Let M^* be a (μ, μ^+)-limit containing M. Suppose that $M \in K_\mu$ is universal over N and $\text{ga-tp}(a/M; M^*)$ does not μ-split over N.

Let M' be an extension of M in K^{am}_μ. Then there exists a $\prec_K$-mapping f such that $f : M' \to M^*$, $f \restriction M = id_M$ and $\text{ga-tp}(a/f(M'))$ does not μ-split over N.

Proof. Notice that M^* is universal over M. So we may assume that M^* contains M'. Since M is universal over N, there exists a $\prec_K$ mapping $h' : M' \to M$ with $h' \restriction N = id_N$. Let $M'' \prec_K M^*$ be an extension of M' such that h' can be extended to $h : M'' \cong M$. Further extend h to h so that $\mathrm{dom}(h) \prec_K M^*$ and $a \in \mathrm{rg}(h)$. By invariance, $\text{ga-tp}(h^{-1}(a)/M'')$ does not μ-split over N.

Subclaim 7.5. $\text{ga-tp}(h^{-1}(a)/M) = \text{ga-tp}(a/M)$.

Proof. We will use the notion of μ-splitting to prove this subclaim. So let us rename the models in such a way that our application of the definition μ-splitting will become transparent. Let $N_1 := M''$ and $N_2 := M$. Let

$p := \text{ga-tp}(h^{-1}(a)/M'')$. Consider the mapping $h: N_1 \cong N_2$. Since p does not μ-split over N, $h(p\restriction N_1) = p\restriction N_2$. Let us calculate this

$$h(p\restriction N_1) = \text{ga-tp}(h(h^{-1}(a))/g(M'')) = \text{ga-tp}(a/M).$$

While,

$$p\restriction N_2 = \text{ga-tp}(h^{-1}(a)/M).$$

Thus $\text{ga-tp}(h^{-1}(a)/M) = \text{ga-tp}(a/M)$ as required. $\dashv$

From the subclaim, we can find a $\prec_K$-mapping g and a model $M^* \prec_K M^\bullet$ such that $g: M'' \to M^*$, $g\restriction M = id_M$ and $g\circ h^{-1}(a) = a$. Notice that $\text{ga-tp}(a/g(M''); M^\bullet)$ does not μ-split over M. $\dashv$

Now we incorporate μ-splitting into our definition of towers.

Definition 7.6.

$$^+\mathcal{K}^*_{\mu,\alpha} := \left\{ (\bar M,\bar a,\bar N) \;\middle|\; \begin{array}{l} (\bar M,\bar a)\in \mathcal{K}^*_{\mu,\alpha};\\ \bar N = \langle N_i \mid i+1<\alpha\rangle;\\ \text{for every } i+1<\alpha,\ N_i \prec_K M_i;\\ M_i \text{ is universal over } N_i \text{ and};\\ \text{ga-tp}(a_i;M_i;M_{i+1}) \text{ does not } \mu\text{-split over } N_i. \end{array} \right\}$$

Similar to the case of $\mathcal{K}^*_{\mu,\alpha}$ we define an ordering,

Definition 7.7. For $(\bar M,\bar a,\bar N)$ and $(\bar M',\bar a',\bar N') \in {}^+\mathcal{K}^*_{\mu,\alpha}$, we say $(\bar M,\bar a,\bar N) <^c_{\mu,\alpha} (\bar M',\bar a',\bar N')$ iff

(1) $(\bar M,\bar a) <^b_{\mu,\alpha} (\bar M',\bar a')$

(2) $\bar N = \bar N'$ and

(3) for every $i<\alpha$, $\text{ga-tp}(a_i/M'_i; M'_{i+1})$ does not μ-split over N_i.

Remark 7.8. Notice that in Definition 7.7, condition (3) follows from (2). We list it as a separate condition to emphasize the role of μ-splitting.

Notation 7.9. We say that $(\bar M,\bar a,\bar N)$ is *nice* iff when i is a limit ordinal $\bigcup_{j<i} M_j$ is an amalgamation base.

The following theorem is a partial solution to a problem from [ShVi]:

Theorem 7.10 (The $<^c_{\mu,\alpha}$-extension property for nice towers). *If $(\bar M,\bar a,\bar N) \in {}^+\mathcal{K}^*_{\mu,\alpha}$ is nice, then there exists a nice $(\bar M',\bar a,\bar N') \in {}^+\mathcal{K}^*_{\mu,\alpha}$ such that $(\bar M,\bar a,\bar N) <^c_{\mu,\alpha} (\bar M',\bar a,\bar N')$.*

Proof. Let μ be a cardinal and α a limit ordinal such that $\alpha < \mu^+ \leq \kappa$. Let $(\bar M,\bar a,\bar N) \in {}^+\mathcal{K}^*_{\mu,\alpha}$ be given. Denote by M_α a model in K^{am}_μ extending $\bigcup_{i<\alpha} M_i$. Fix $M^\bullet$ to be a (μ,μ^+)-limit model over M_α.

We will define by induction on $i<\alpha$ a sequence of models $\langle M'_i \mid i<\alpha\rangle$ and sequences of $\prec_K$-mappings $\langle f'_{j,i} \mid j<i<\alpha\rangle$ and $\langle f^\bullet_{j,i} \mid j<i<\alpha\rangle$ such that for $i\leq\alpha$:

(1) $(\langle f'_{j,i}(M'_j) \mid j\leq i\rangle, \bar a\restriction i, \bar N\restriction i)$ is a $<^c_{\mu,i}$-extension of $(\bar M,\bar a,\bar N)\restriction i$,

(2) $(\langle M'_j \mid j<i\rangle, \langle f'_{j,i} \mid j\leq i\rangle)$ forms a directed system,

(3) M'_i is universal over M_i,
(4) M'_{i+1} is universal over $f'_{i,i+1}(M'_i)$,
(5) $f'_{j,i}\restriction M_j = \mathrm{id}_{M_j}$,
(6) $M'_i \prec_K M^*$,
(7) $f'_{j,i}$ can be extended to an automorphism of M^*, $f^*_{j,i}$, for $j \leq i$ and
(8) $(\langle M^* \mid j < i\rangle, \langle f^*_{k,j} \mid k \leq j < i\rangle)$ forms a directed system.

The construction is enough: We can take M'_δ and $\langle f'_{i,\delta} \mid i < \delta\rangle$ to be a direct limit of $(\langle M'_i \mid i < \delta\rangle, \langle f'_{j,i} \mid j \leq i < \delta\rangle)$. Since $f'_{j,i}\restriction M_j = \mathrm{id}_{M_j}$, for every $j \leq i < \delta$, we may assume that $f'_{i,\delta}\restriction M_i = \mathrm{id}_{M_i}$ for every $i < \delta$. Notice that $(\langle f'_{i,\delta}(M'_i) \mid i < \delta\rangle, \bar{a})$ is a $<^c_{\mu,\delta}$-extension of $(\bar{M}, \bar{a})$.

The construction is possible:

$i = 0$: Since M_0 is an amalgamation base, we can find $M''_0 \in K^*_\mu$ (a first approximation of the desired M'_0) such that M''_0 is universal over M_0. By Theorem 7.4, we may assume that ga-tp(a_0/M''_0) does not μ-split over N_0 and $M''_0 \prec_K M^*$. Since $a_0 \notin M_0$ and ga-tp(a_0/M_0) does not μ-split over N_0, we know that $a_0 \notin M''_0$. But, we might have that for some $l > 0$, $a_l \in M''_0$. We use weak disjoint amalgamation to avoid $\{a_l \mid 0 < l < \delta\}$. By the Downward Löweneim-Skolem Axiom for AECs (Axiom 6) we can choose $M^2 \in K_\mu$ such that $M''_0, M_1 \prec_K M^2 \prec_K M^*$.

By Corollary 5.3 (applied to M_1, M_δ, M^2 and $\langle a_l \mid 0 < l < \delta\rangle$), we can find a $\prec_K$-mapping f such that

- $f : M^2 \to M^*$
- $f\restriction M_1 = \mathrm{id}_{M_1}$
- $f(M^2) \cap \{a_l \mid 0 < l < \delta\} = \emptyset$

Define $M'_0 := f(M''_0)$. Notice that $a_0 \notin M'_0$ because $a_0 \notin M''_0$ and $f(a_0) = a_0$. Clearly $M'_0 \cap \{a_l \mid 0 \leq l < \delta\} = \emptyset$, since $M''_0 \prec_K M^2$ and $f(M^2) \cap \{a_l \mid 0 < l < \delta\} = \emptyset$. We need only verify that ga-tp(a_0/M'_0) does not μ-split over N_0. By invariance, ga-tp(a_0/M''_0) does not μ-split over N_0 implies that ga-tp$(f(a_0)/f(M''_0))$ does not μ-split over N_0. But recall $f(a_0) = a_0$ and $f(M''_0) = M'_0$. Thus ga-tp(a_0/M'_0) does not μ-split over N_0.

Set $f^*_{0,0} := \mathrm{id}_{M^*}$ and $f'_{0,0} := \mathrm{id}_{M'_0}$.

$i = j + 1$: Suppose that we have completed the construction for all $k \leq j$. Since $M'_j, M_{j+1} \prec_K M^*$, we can apply the Downward-Löwenheim Axiom for AECs to find M'''_{j+1} (a first approximation to M'_{j+1}) a model of cardinality μ extending both M'_j and M_{j+1}. WLOG by Subclaim 6.10 we may assume that M'''_{j+1} is a limit model of cardinality μ, and M'''_{j+1} is universal over M_{j+1} and M'_j. By Theorem 7.4, we can find a $\prec_K$ mapping $f : M'''_{j+1} \to M^*$ such that $f\restriction M_{j+1} = \mathrm{id}_{M_{j+1}}$ and ga-tp$(a_{j+1}/f(M'''_{j+1}))$ does not μ-split over N_{j+1}. Set $M''_{j+1} := f(M'''_{j+1})$.

Subclaim 7.11. $a_{j+1} \notin M''_{j+1}$

Proof. Suppose that $a_{j+1} \in M''_{j+1}$. Since M'_{j+1} is universal over N_{j+1}, there exists a $\prec_K$-mapping $g : M''_{j+1} \to M_{j+1}$ such that $g\restriction N_{j+1} = \mathrm{id}_{N_{j+1}}$. Since

$\text{ga-tp}(a_{j+1}/M''_{j+1})$ does not μ-split over N_{j+1}, we have that

$$\text{ga-tp}(a_{j+1}/g(M''_{j+1})) = \text{ga-tp}(g(a_{j+1})/g(M''_{j+1})).$$

Notice that because $g(a_{j+1}) \in g(M''_{j+1})$, we have that $a_{j+1} = g(a_{j+1})$. Thus $a_{j+1} \in g(M''_{j+1}) \prec_K M_{j+1}$. This contradicts the definition of towers $a_{j+1} \notin M_{j+1}$.

$\dashv$

M''_{j+1} may serve us well if it does not contain any a_l for $j+1 \leq l < \delta$, but this is not guaranteed. So we need to make an adjustment. Let M^2 be a model of cardinality μ such that $M_{j+2}, M''_{j+1} \prec_K M^2 \prec_K M^*$. Notice that M^* is universal over M_{j+2}. Thus we can apply Corollary 5.3 to M_{j+2}, M_δ, M^2 and $\langle a_l \mid j+2 \leq l < \delta\rangle$. This yields a $\prec_K$-mapping h such that

- $h : M^2 \to M^*$
- $h \restriction M_{j+2} = \mathrm{id}_{M_{j+2}}$ and
- $h(M^2) \cap \{a_l \mid j+2 \leq l < \delta\} = \emptyset$.

Set $M'_{j+1} := h(M''_{j+1})$. Notice that by invariance $\text{ga-tp}(a_{j+1}/M''_{j+1})$ does not μ-split over N_{j+1} implies that $\text{ga-tp}(h(a_{j+1})/h(M''_{j+1}))$ does not μ-split over $h(N_{j+1})$. Recalling that $h \restriction M_{j+2} = \mathrm{id}_{M_{j+2}}$ we have that $\text{ga-tp}(a_{j+1}/M''_{j+1})$ does not μ-split over N_{j+1}. We need to verify that $a_{j+1} \notin M'_{j+1}$. This holds because $a_{j+1} \notin M''_{j+1}$ and $h(a_{j+1}) = a_{j+1}$.

Set $f'_{j+1,j+1} = \mathrm{id}_{M_{j+1,j+1}}$ and $f^*_{j+1,j+1} = \mathrm{id}_{M^*}$ and $f'_{j,j+1} := h \circ f \restriction M'_j$. Since M^* is a (μ,μ^+)-limit over both M'_j and $f'_{j,j+1}(M'_j)$, we can extend $f'_{j,j+1}$ to an automorphism of M^*, denoted by $f^*_{j,j+1}$.

To guarantee that we have a directed system, for $k < j$, define $f'_{k,j+1} := f'_{j,j+1} \circ f'_{k,j}$ and $f^*_{k,j+1} := f^*_{j,j+1} \circ f^*_{k,j}$.

i is a limit ordinal. Suppose that $(\langle M'_j \mid j < i\rangle, \langle f'_{k,j} \mid k \leq j < i\rangle)$ and $(\langle M^* \mid j < i\rangle, \langle f^*_{k,j} \mid k \leq j < i\rangle)$ have been defined. Since they are both directed systems, we can take direct limits. By niceness we can apply Claim 6.11 so that we may assume that $(\acute{M}_i, \langle \acute{f}_{j,i} \mid j < i\rangle)$ and $(M^*, \langle f^{*}_{j,i} \mid j < i\rangle)$ are the respective direct limits such that $\acute{M}_i \prec_K M^*$ and $\bigcup_{j<i} M_j \prec_K \acute{M}_i$. By Condition (4) of the construction, notice that $\acute{M}_i$ is a (μ, i)-limit model witnessed by $\langle \acute{f}_{j,i}(M'_j) \mid j < i\rangle$. Hence $\acute{M}_i$ is an amalgamation base. Since $\acute{M}_i$ and M_i both lie inside of M^*, we can find $M'''_i \in \acute{K}_\mu$ which is universal over M_i and universal over $\acute{M}_i$.

By Theorem 7.4 we can find a $\prec_K$-mapping $f : M'''_i \to M^*$ such that $f \restriction M_i = \mathrm{id}_{M_i}$ and $\text{ga-tp}(a_i/f(M'''_i))$ does not μ-split over N_i. Set $M''_i := f(M'''_i)$. By a similar argument to Subclaim 7.11 we can see that $a_i \notin M''_i$.

M''_i may contain some a_l when $i \leq l < \delta$. We need to make an adjustment using weak disjoint amalgamation. Let M^2 be a model of cardinality μ such that $M''_i, M_{i+1} \prec_K M^2 \prec_K M^*$. By Corollary 5.3 applied to M_i, M_δ, M^2 and $\langle a_l \mid i < l < \delta\rangle$ we can find $h : M''_i \to M^*$ such that $h \restriction M_{i+1} = \mathrm{id}_{M_{i+1}}$ and $h(M^2) \cap \{a_l \mid i < l < \delta\} = \emptyset$.

Set $M^0_i := h(M^{00}_i)$. We need to verify that $a_i \notin M^0_i$ and ga-tp(a_i/M^0_i) does not μ-split over N_i. Since $a_i \notin M^{00}_i$ and $h(a_i) = a_i$, we have that $a_i \notin h(M^{00}_i) = M^0_i$. By invariance of non-splitting, ga-tp(a_i/M^{00}_i) not μ-splitting over N_i implies that ga-tp$(h(a_i)/h(M^{00}_i))$ does not μ-split over $h(N_i)$. Recalling our definition of h and M^0_i. This yields ga-tp(a_i/M^0_i) does not μ-split over N_i.

Set $f^0_{i,i} := id_{M_i}$, $f^*_{i,i} := id_{M^*}$ and for $j < i$, $f^0_{j,i} := h \circ f \circ f'_{j,i}$.

Notice that for every $j < i$, M^* is a (μ,μ^+)-limit over both M^0_j and $f^0_{j,i}(M^0_j)$. Thus by the uniqueness of (μ,μ^+)-limit models, we can extend $f^0_{j,i}$ to an automorphism of M^*, denoted by $f^*_{j,i}$.

$\dashv$

8. Scattered Towers

We now make the final modification to the towers and prove an extension theorem for these scattered towers. Let's recall the general strategy for proving the uniqueness of limit models. Our goal is to construct an array of models $\langle M^i_j \mid j \leq \theta_1, i \leq \theta_2\rangle$ of width θ_1 and height θ_2 such that the union will be simultaneously a (μ,θ_1)-limit model (witnessed by $\langle M^{\theta_2}_j \mid j < \theta_1\rangle$) and a (μ,θ_2)-limit model (witnessed by $\langle M^i_{\theta_1} \mid i < \theta_2\rangle$). In spirit our construction will behave this way. However, such a construction is too much to hope for because:

(1) Given a tower $(\bar{M},\bar{a},\bar{N}) \in K^*_{\mu,\theta_1}$, there is no reason to believe that $\bigcup_{i<\theta_1} M_i$ is a (μ,θ_1)-limit model since we do not require that M_{i+1} is universal over M_i, nor do we require that this chain of models is continuous. In fact, we cannot conclude that $\bigcup_{i<\theta_1} M_i$ is even an amalgamation base.

(2) While our ordering on towers is enough to get that $M^{\theta_2}_i$ is a (μ,θ_2)-limit for $i < \theta_1$ (witnessed by $\langle M^j_i \mid j < \theta_2\rangle$), we cannot say anything about the model $M^{\theta_2}_{\theta_1}$. Unfortunately it is not reasonable to "fix" our definition of ordering to guarantee that $M^{\theta_2}_{\theta_1}$ is a limit model, since we would then be unable (at least we see no way of doing it) to prove the extension property for towers.

In Section 9, we address problem (1) by identifying some properties of towers (full and reduced) that guarantee that the top of the tower ($M^{\theta_2}_{\theta_1}$) is in fact a (μ,θ_1)-limit model.

To remedy (2) we define scattered towers. Since we know that $M^{\theta_2}_i$ is a (μ,θ_2)-limit for $i < \theta_1$ (witnessed by $\langle M^j_i \mid j < \theta_2\rangle$), the idea is to construct a very wide array of towers (of width μ^+) and then focus in on some $\alpha < \mu^+$ of cofinality θ_1. Then $M^{\theta_2}_\alpha$ won't be in the last column of the array, so the ordering will guarantee us that $M^{\theta_2}_\alpha$ is a (μ,θ_2)-limit (witnessed by $\langle M^j_\alpha \mid j < \theta_2\rangle$). However, we have not proved an extension property for towers of width μ^+. Our arguments won't generalize to K^*_{μ,μ^+} because Theorem 5.1 (Weak Disjoint Amalgamation) isn't strong enough since we would have μ^+ many elements to avoid ($\{a_i \mid i < \mu^+\}$). So we will construct the tower in K^*_{μ,μ^+} in μ^+-many stages by shorter towers (in $K^*_{\mu,\alpha}$ for $\alpha < \mu^+$). To do this we introduce the notion of scattered towers, which will allow us to extend a tower in $K^*_{\mu,\alpha}$ to a longer tower in $K^*_{\mu,\beta}$ when $\alpha < \beta < \mu^+$.

Notation 8.1. Let α be an ordinal. We say that $U \subseteq P(\alpha)$ is a set of intervals of α provided that

- $0 \in \bigcup U$,
- for $u_1 \neq u_2 \in U$, $u_1 \cap u_2 = \emptyset$ and
- for $u \in U$, if $\beta_1 < \beta_2 \in u$, then for every γ with $\beta_1 < \gamma < \beta_2$, we have $\gamma \in u$.

Definition 8.2 (Definition 3.3.1 of [ShVi]). For U a set of intervals of ordinals $< \mu^+$, let

$$^+K^*_{\mu,U} := \left\{ (\bar{M},\bar{a},\bar{N}) \;\middle|\; \begin{array}{l} \bar{M} = \langle M_i \mid i \in u \text{ for some interval } u \in U\rangle; \\ \bar{M} \text{ is } \prec_K \text{ increasing, but not} \\ \text{necessarily continuous}; \\ a_i \in M_{i+1}\setminus M_i \text{ when } i, i+1 \in \bigcup U; \\ \bar{N} = \langle N_i \mid i \in \bigcup U\rangle; \\ M_i \text{ is universal over } N_i \text{ when } i \in \bigcup U \text{ and} \\ \text{ga-tp}(a_i/M_i, M_{i+1}) \text{ does not } \mu\text{-split over } N_i \end{array} \right\}.$$

Notice that these scattered towers are in some sense subtowers of the towers $^+K^*_{\mu,\alpha}$. Hence we can consider the restriction of $\leq^c_{\mu,\alpha}$ to the class $^+K^*_{\mu,U}$:

Definition 8.3 (Definition 3.3.2 of [ShVi]). Let $(\bar{M}^l,\bar{a}^l,\bar{N}^l) \in {}^+K^*_{\mu,U}$ for $l = 1,2$. $(\bar{M}^1,\bar{a}^1,\bar{N}^1) \leq^c (\bar{M}^2,\bar{a}^2,\bar{N}^2)$ iff for $i \in \bigcup U$,

(1) $M^1_i \preceq_K M^2_i$, $a^1_i = a^2_i$ and $N^1_i = N^2_i$ and
(2) if $M^1_i \neq M^2_i$, then M^2_i is universal over M^1_i.

We say that $(\bar{M}^1,\bar{a}^1,\bar{N}^1) <^c (\bar{M}^2,\bar{a}^2,\bar{N}^2)$ provided that for every $i \in \bigcup U$, $M^1_i \neq M^2_i$.

Actually we can extend the ordering to compare towers from classes $^+K^*_{\mu,U_1}$ and $^+K^*_{\mu,U_2}$ when U_2 is an interval-extension of U_1. By interval-extension we mean:

Definition 8.4. U_2 is an interval-extension of U_1 iff for every $u_1 \in U_1$, there is $u_2 \in U_2$ such that $u_1 \subseteq u_2$.

Definition 8.5. Let $(\bar{M}^l,\bar{a}^l,\bar{N}^l) \in {}^+K^*_{\mu,U_l}$ for $l = 1,2$. $(\bar{M}^1,\bar{a}^1,\bar{N}^1) \leq^c (\bar{M}^2,\bar{a}^2,\bar{N}^2)$ iff for $i \in \bigcup U_1$,

(1) $M^1_i \preceq_K M^2_i$, $a^1_i = a^2_i$ and $N^1_i = N^2_i$ and
(2) if $M^1_i \neq M^2_i$, then M^2_i is universal over M^1_i.

Now we can generalize the notion of niceness and prove an extension property for the class of all scattered towers.

Definition 8.6. A scattered tower $(\bar{M}, \bar{a}, \bar{N}) \in {}^+K^*_{\mu,U}$ is said to be nice provided that whenever δ a limit ordinal is a limit of some sequence of elements from $\bigcup U$, then $\bigcup_{j \in \bigcup U, j<i} M_j$ is an amalgamation base.

Theorem 8.7 ($<^c$-Extension Property for Nice Scattered Towers). Let U^1 and U^2 be sets of intervals of ordinals $< \mu^+$ such that U^2 is an interval extension of U^1. Let $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \in {}^+K^*_{\mu,U^1}$ be a nice scattered tower. There exists a nice scattered tower $(\bar{M}^2, \bar{a}^2, \bar{N}^2) \in {}^+K^*_{\mu,U^2}$ such that $(\bar{M}^1, \bar{a}^1, \bar{N}^1) <^c (\bar{M}^2, \bar{a}^2, \bar{N}^2)$.

Proof. WLOG we can rewrite U^2 as collection of disjoint intervals such that for every $u^2 \in U^2$, there exists at most one $u^1 \in U^1$ such that $u^1 \subseteq u^2$. Let us enumerate U^1 as $\langle u^1_t \mid t \in \alpha^1 \rangle$ in increasing order (in other words when $t < t' \in \alpha^1$ we have that $\max(u^1_t) < \min(u^1_{t'})$.)

For bookkeeping purposes, we will enumerate U^2 as $\langle u^2_t \mid t \in \alpha^1 \rangle$ as

$$u^2_t = \begin{cases} \{i \in \bigcup U^2 \mid \min\{u^1_t\} \leq i < \min\{u^1_{t+1}\}\} & \text{if } t+1 < \alpha^1 \\ \{i \in \bigcup U^2 \mid \min\{u^1_t\} \leq i\} & \text{otherwise} \end{cases}$$

Since $0 \in \bigcup U^1$, this enumeration can be carried out.

Given $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \in {}^+K^*_{\mu,U^1}$ a nice tower, we will find a $<^c$-extension in ${}^+K^*_{\mu,U^2}$ by using direct limits inside a (μ,μ^+)-limit model as we have done in the proofs of Theorem 6.8 and Theorem 7.10. As before, fix M^* a (μ,μ^+)-limit model containing $\bigcup_{i \in \bigcup U^1} M^1_i$. We will define approximations to a tower in ${}^+K^*_{\mu,U^2}$ with towers in ${}^+K^*_{\mu,U^2_t}$ extending towers in ${}^+K^*_{\mu,U^1_t}$ where $U^1_t = \{u^1_s \mid s \leq t\}$ for $l = 1, 2$.

These partial extensions will be defined by constructing sequences of models $\langle M^2_i \mid i \in \bigcup U^2 \rangle$ and $\langle N^2_i \mid i+1 \in \bigcup U^2 \rangle$, a sequence of elements $\langle a^2_i \mid i+1 \in \bigcup U^2 \rangle$ and $\preceq_K$-mappings $\{f_{i,j} \mid i \leq j \leq \alpha^1\}$ satisfying

1. $(\langle f_{s,t}(M^2_i) \mid i \in u^2_s \text{ and } s \leq t \rangle, \bar{a}^t, \bar{N}^t)$ is a $<^c_{\mu,U^1_t}$-extension of $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \restriction U^1_t$ where $\bar{a}^t = \langle a^2_i \mid i+1 \in U^2_t \rangle$ and $\bar{N}^t = \langle N^2_i \mid i+1 \in U^2_t \rangle$,
2. $(\langle M^s \mid s \leq t \rangle, \langle f_{s,t} \mid s \leq t \rangle)$ forms a directed system where $M^s = \bigcup_{i \in u^2_s} M^2_i$.
3. M^2_i is universal over M^1_i for all $i \in \bigcup U^2_t$,
4. M^2_j is universal over $f_{s,t}(M^2_i)$ for every $i < j$ and $s \leq t$ such that $i \in u^2_s$ and $j \in u^2_t$ (consequently M^{t+1} is universal over $f_{t,t+1}(M^t)$),
5. $f_{s,t} \restriction M^1_j = id_{M^1_j}$ for all $j \in u^2_s$,
6. $M^2_i \preceq_K M^*$,
7. $f_{s,t}$ can be extended to an automorphism of M^*, $f^*_{s,t}$, for $s \leq t < \alpha^1$ and
8. $(\langle M^* \mid s \leq t \rangle, \langle f^*_{s,t} \mid s \leq t \rangle)$ forms a directed system.

The construction:

$t=0$: First notice that by Theorem 7.10, we can find $\langle M'_i \mid i\in u_0^1\rangle$ such that $(\bar M', \bar a^1\restriction u_0^1, \bar N^1\restriction u_0^1)$ is a $<^c_{U_0^1}$-extension of $(\bar M^1,\bar a^1,\bar N^1)\restriction U_0^1$ and $\bar M'$ avoids $\bar a^1$ above u_0^1 (specifically $(\bigcup_{i\in u_0^1} M'_i) \bigcap \{a^j \mid j\in \bigcup U^1\setminus u_0^1\} = \emptyset$.) Moreover the proof of Theorem 7.10 gives us an extension such that $\bigcup_{i\in u_0^1} M'_i$ is a limit model.

We can choose $M^\dagger \in K_\lambda$ such that $\bigcup_{i\in u_0^1} M'_i, M^1_{\min\{u_1^1\}} \prec_K M^\dagger \prec_K M^*$ and $M^\dagger$ is a $(\lambda, |u_0^2|+\aleph_0)$-limit over $\bigcup_{i\in u_0^1} M'_i$. This is possible since $\bigcup_{i\in u_0^1} M'_i$ is an amalgamation base. Let $\langle M^\dagger_j \mid j < |u_0^2|+\aleph_0\rangle$ witness that $M^\dagger$ is a $(\lambda,|u_0^2|+\aleph_0)$-limit over $\bigcup_{i\in u_0^1} M'_i$. Since limit models are amalgamation bases, we may choose $M^\dagger_{+1}$ to be a (λ,ω)-limit over $M^\dagger$.

By Corollary 5.3 applied to $(\bigcup_{i\in u_0^1} M_i^1, \bigcup_{i\in u_0^1} M'_i, M^\dagger)$ and $\{a_j^1 \mid j\in \bigcup U^1\setminus U_0^1\}$, there exists an automorphism g of M^* such that

- $g\restriction \bigcup_{i\in u_0^1} M_i^1 = id_{\bigcup_{i\in u_0^1} M_i^1}$ and
- $g(M^\dagger)\cap \{a_j^1 \mid j\in \bigcup U^1\setminus u_0^1\}=\emptyset$.

Denote by $\langle i_j \mid j \in otp(u_0^2\setminus u_0^1)\rangle$ the increasing enumeration of $u_0^2\setminus u_0^1$. Define

$$M_i^2 := \begin{cases} g(M'_i) & \text{for } i\in u_0^1 \\ g(M^\dagger_j) & \text{for } i=i_j\in u_0^2\setminus u_0^1 \end{cases}$$

Since $M^\dagger$ is an limit model witnessed by the $M^\dagger_j$'s, we can choose $a_i \in M^2_{i+1}\setminus M_i^2$ for all $i, i+1\in u_0^2\setminus u_0^1$. Since M_i^2 is a limit model for each $i, i+1\in u_0^2\setminus u_0^1$, we can apply Theorem 7.2 to find $N_i^2 \prec_K M_i^2$ such that ga-tp(a_i/M_i^2) does not λ-split over N_i^2 and M_i^2 is universal over N_i^2.

All that remains is to define $f_{0,0} := id_{\bigcup_{i\in u_0^1} M_i^1}$ and $f^*_{0,0} := id_{M^*}$.

$t=s+1$: By condition (4) of the construction, we have that $\bigcup_{i\in u_s^2} M_i^2$ is a limit model witnessed by $\langle f_{r,s}(M_i^2) \mid i\in u_r^2$ and $r\leq s\rangle$. Thus $\bigcup_{i\in u_s^2} M_i^2$ is an amalgamation base. Now we can choose a model $M'\in K_\lambda$ such that $\bigcup_{i\in u_s^2} M_i^2, M^1_{\min\{u^1_{s+1}\}} \prec_K M'$ and M'' is a $(\lambda,|u^2_{s+1}|+\aleph_0)$-limit over $\bigcup_{i\in u_s^2} M_i^2$. By identical arguments to the successor case in Theorem 7.10, we can find $\bar M' = \langle M'_i \mid i\in U_s^2 \bigcup u^1_{s+1}\rangle$ and an automorphism h of M^* such that

- $(\bar M',\bar a',\bar N')$ is a nice scattered tower, where $\bar a' = \langle a_i^2 \mid i\in U_s^2\bigcup u^1_{s+1}\rangle$ and $\bar N' = \langle N_i^2 \mid i\in U_s^2\bigcup u^1_{s+1}\rangle$
- $(\bar M^1,\bar a^1,\bar N^1)\restriction U^1_{s+1} <^c (\bar M',\bar a',\bar N')$
- $\bigcup_{i\in U_s^2\bigcup u^1_{s+1}} M'_i \cap \{a_j^1 \mid j\in U^1\setminus U^1_{s+1}\}=\emptyset$.
- $h\restriction M'': M''\cong M'_{\min\{u^1_{s+1}\}}$ and
- $h\restriction M^1_{\min\{u^1_{s+1}\}} = id_{M^1_{\min\{u^1_{s+1}\}}}$.

Let $M^{\dagger}$ be a $(\lambda, |u^2_{s+1} \setminus u^1_{s+1}| + \aleph_0)$-limit model over $\bigcup_{i \in U^2_s \bigcup u^1_{s+1}} M^0_i$ such that such that $M^1_{\min\{u^2_{s+2}\}} \leq_K M^{\dagger} \leq_K M^*$. Let $\langle M^{\dagger}_j \mid < |u^2_{s+1} \setminus u^1_{s+1}| + \aleph_0 \rangle$ witness that $M^{\dagger}$ is a limit model. Since limit models are amalgamation bases, we may choose $M^{\dagger}_{+1}$ to be a (λ, ω)-limit over $M^{\dagger}$.

Applying Corollary 5.3 to $(\bigcup_{i \in u^1_{s+1}} M^1_i, \bigcup_{i \in U^2_s \bigcup u^1_{s+1}} M^0_i, M^{\dagger})$ and $\{a^1_j \mid j \in \bigcup U^1 \setminus U^1_{s+1}\}$, there exists an automorphism of M^*, g, such that

- $g \restriction \bigcup_{i \in u^1_{s+1}} M^1_i = id_{\bigcup_{i \in u^1_{s+1}} M^1_i}$ and
- $g(M^{\dagger}) \cap \{a^1_j \mid j \in \bigcup U^1 \setminus U^1_{s+1}\} = \emptyset$.

Denote by $\langle i_j \mid j \in otp(u^2_{s+1} \setminus u^1_{s+1}) \rangle$ the increasing enumeration of $u^2_{s+1} \setminus u^1_{s+1}$. Define

$$M^2_i := \begin{cases} g(M^0_i) & \text{for } i \in u^1_{s+1} \\ g(M^{\dagger}_j) & \text{for } i = i_j \in u^2_{s+1} \setminus u^1_{s+1} \end{cases}$$

Since $M^{\dagger}$ is a limit model witnessed by the $M^{\dagger}_j$'s, we can choose $a_i \in M^2_{i+1} \setminus M^2_i$ for all $i, i+1 \in u^2_{s+1} \setminus u^1_{s+1}$. Since M^2_i is a limit model for each $i, i+1 \in u^2_{s+1} \setminus u^1_{s+1}$, we can apply Theorem 7.2 to find $N^2_i \restriction_K M^2_i$ such that ga-tp(a_i/M^2_i) does not λ-split over N^2_i and M^2_i is universal over N^2_i.

Define $f_{s,s+1} := g - h \restriction \bigcup_{i \in u^2_s} M^2_i$ and $f^*_{s,s+1} := g - h$. To complete the definition of a directed system, for every $r \leq s$, set $f_{r,s+1} := f_{s,s+1} - f_{r,s}$ and $f^*_{r,s} := f^*_{s,s+1} - f^*_{r,s}$.

t is a limit ordinal: Suppose that $\langle h \bigcup_{i \in u^2_s} M^2_i (= M^s) \mid s < t \rangle, \langle f_{r,s} \mid r \leq s < t \rangle$ and $\langle M^* \mid s < t \rangle, \langle f^*_{r,s} \mid r \leq s < t \rangle$ have been defined. Since these are both directed systems, we can take direct limits. By niceness, we can apply Claim 6.11 so that we may assume that $(M', \langle f'_{s,t} \mid s \leq t \rangle)$ and $(M^*, \langle f^{*\prime}_{s,t} \mid s \leq t \rangle)$ are the respective direct limits such that $M' \leq_K M^*$, $f^{*\prime}_{s,t} \quad f'_{s,t}$ and $\bigcup_{s<t} \bigcup_{i \in u^1_s} M^1_i \leq_K M'$.

By condition (4) of the construction, notice that M' is a (λ, t)-limit model witnessed by $\langle f'_{s,t}(M^s) \mid s < t \rangle$. Hence M'_t is an amalgamation base. As in the successor case of the construction in the proof of Theorem 7.10, we can find $\bar{M}^0 = \langle M^0_i \mid i \in \bigcup_{s<t} U^2_s \bigcup u^1_t \rangle$ and an automorphism of M^* such that

- $(\bar{M}^0, \bar{a}^0, \bar{N}^0)$ is a nicely scattered over, where $\bar{a}^0 = \langle a^2_i \mid i \in \bigcup_{s<t} U^2_s \bigcup u^1_t \rangle$ and $\bar{N}^0 = \langle N^2_i \mid i \in \bigcup_{s<t} U^2_s \bigcup u^1_t \rangle$
- $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \restriction U^1_t <^c (\bar{M}^0, \bar{a}^0, \bar{N}^0)$
- $\bigcup_{i \in \bigcup_{s<t} U^2_s \bigcup u^1_t} M^0_i \cap \{a^1_j \mid j \in U^1 \setminus U^1_t\} = \emptyset$.
- $h \restriction M' : M' \cong M^0_{\min\{u^1_t\}}$ and
- $h \restriction M^1_{\min\{u^1_t\}} = id_{M^1_{\min\{u^1_t\}}}$.

Let $M^{\dagger}$ be a $(\lambda, |u^2_t \setminus u^1_t| + \aleph_0)$-limit model over $\bigcup_{i \in \bigcup_{s<t} U^2_s \bigcup u^1_t} M^0_i$ such that such that $M^1_{\min\{u^2_{t+1}\}} \leq_K M^{\dagger} \leq_K M^*$. Let $\langle M^{\dagger}_j \mid < |u^2_{s+1} \setminus u^1_{s+1}| + \aleph_0 \rangle$

witness that $M^{\dagger}$ is a limit model. Since limit models are amalgamation bases, we may choose $M^{\dagger}_{+1}$ to be a (μ,ω)-limit over $M^{\dagger}$.

Applying Corollary 5.3 to $(\bigcup_{i\in u^1_t} M^1_i, \bigcup_{i\in\bigcup_{s<t}U^2_s\cup u^1_t} M^0_i, M^{\dagger})$ and $\{a^1_j \mid j\in\bigcup U^1\setminus U^1_t\}$, there exists an automorphism of $\check{M}$, g, such that

- $g\restriction\bigcup_{i\in u^1_t} M^1_i = id_{\bigcup_{i\in u^1_t} M^1_i}$ and
- $g(M^{\dagger})\cap\{a^1_j \mid j\in\bigcup U^1\setminus U^1_t\}=\emptyset$.

Denote by $\langle i_j \mid j\in otp(u^2_t\setminus u^1_t)\rangle$ the increasing enumeration of $u^2_t\setminus u^1_t$. Define

$$M^2_i := \begin{cases} g(M^0_i) & \text{for } i\in u^1_t \\ g(M^{\dagger}) & \text{for } i=i_j\in u^2_t\setminus u^1_t \end{cases}$$

Since $M^{\dagger}$ is a limit model witnessed by the $M^{\dagger}$'s, we can choose $a_i\in M^2_{i+1}\setminus M^2_i$ for all $i,i+1\in u^2_t\setminus u^1_t$. Since M^2_i is a limit model for each $i,i+1\in u^2_t\setminus u^1_t$, we can apply Theorem 7.2 to find $N^2_i\prec_K M^2_i$ such that ga-tp(a_i/M^2_i) does not μ-split over N^2_i and M^2_i is universal over N^2_i.

Define $f_{s,t}:=g\circ h\circ\acute{f}_{s,t}\restriction\bigcup_{i\in u^2_s}M^2_i$ and $\check{f}_{s,t}:=g\circ h\circ\acute{f}_{s,t}$ for all $s<t$. $\dashv$

Notice that in the proof of the $<^c$-extension property for nice towers, we have actually shown that there is some freedom in choosing the new a^0_is:

Corollary 8.8. Let U^1 and U^2 be sets of intervals of ordinals $<\mu^+$ such that U^2 is an interval extension of U^1. Let $(\bar{M}^1,\bar{a}^1,\bar{N}^1)\in{}^+K^*_{\mu,U^1}$ be a nice scattered tower. There exists a nice scattered tower $(\bar{M}^2,\bar{a}^2,\bar{N}^2)\in{}^+K^*_{\mu,\bigcup U^2}$ such that $(\bar{M}^1,\bar{a}^1,\bar{N}^1)<^c(\bar{M}^2,\bar{a}^2,\bar{N}^2)$. Moreover for every $i\in\bigcup U^2\setminus\bigcup U^1$ and every $j<i$ with $j\in U^1$, if $(p,N)\in\mathfrak{St}(M^1_j)$, then we can choose a_i such that $(p,N)\sim(\text{ga-tp}(a^2_i/M^2_i),N^2_i)$.

9. Reduced and Full Towers

In this section we identify a property (full and continuous) which will guarantee that for a tower $(\bar{M},\bar{a},\bar{N})\in K^*_{\mu,\alpha}$ with this property, we have that $\bigcup_{i<\alpha}M_i$ is a (μ,α)-limit model over M_0 (see Theorem 9.10). This addresses problem (1) in our construction of an array of models described at the beginning of Section 8. The first point that (1) breaks down is that $\langle M^2_i \mid i<\delta_1\rangle$ need not be a continuous chain of models, since we do not require towers to be continuous. Shelah introduced the following concept which captures some continuous towers (see Theorem 9.5). It was originally introduced in [ShVi] for ${}^+K^*_{\mu,\alpha}$ towers. We generalize it to the context of scattered towers:

Definition 9.1. A tower $(\bar{M},\bar{a},\bar{N}) \in {}^{+}K^{*}_{\mu,U}$ is said to be *reduced* provided that for every $(\bar{M}',\bar{a},\bar{N}) \in {}^{+}K^{*}_{\mu,U}$ with $(\bar{M},\bar{a},\bar{N}) \leq^{c} (\bar{M}',\bar{a},\bar{N})$ we have that for every $i \in \bigcup U$,

$$M'_i \cap \bigcup_{j\in\bigcup U} M_j = M_i.$$

Notice that for a tower in ${}^{+}K^{*}_{\mu,U}$ to be reduced, it only depends on other towers in ${}^{+}K^{*}_{\mu,U}$ (towers with the same index set). However, if we take a $<^{c}$-increasing and continuous chain of reduced towers with increasing index sets, the union will be reduced:

Proposition 9.2. Let $\langle U_\gamma \mid \gamma < \beta\rangle$ be an increasing and continuous sequence of sets of intervals ($\bigcup U_{\gamma+1}$ is an interval-extension of $\bigcup U_\gamma$ and if β is a limit ordinal $\bigcup U_\beta = \bigcup_{\gamma<\beta} U_\gamma$.) If $\langle(\bar{M},\bar{a},\bar{N})^{\gamma} \mid \gamma < \beta\rangle$ is $<^{c}$-increasing and continuous sequence of reduced towers, then the union of these towers is reduced.

Proof. Denote by $(\bar{M},\bar{a},\bar{N})^{\beta}$ the limit of these sequence of towers and U_β the limit of their intervals. More specifically, U_β is a fixed set of intervals such that $\bigcup U_\beta = \bigcup_{\gamma<\beta}\bigcup U_\gamma$ and for every $\gamma<\beta$, U_β is an interval extension of U_γ. $\bar{M}^{\beta} = \langle M^{\beta}_i \mid i \in \bigcup U_\beta\rangle$ where $M^{\beta}_i = \bigcup_{\{\gamma<\beta \mid i\in\bigcup U_\gamma\}} M^{\gamma}_i$. $\bar{N}^{\beta} = \langle N_i^{\min\{\gamma \mid i\in\bigcup U_\gamma\}} \mid i \in \bigcup U_\beta\rangle$ and $\bar{a}^{\beta} = \langle a_i^{\min\{\gamma \mid i\in\bigcup U_\gamma\}} \mid i \in \bigcup U_\beta\rangle$.

Suppose that it is not reduced. Let $(\bar{M}',\bar{a},\bar{N}) \in {}^{+}K^{*}_{\mu,U_\beta}$ witness this. Then there exists an $i \in \bigcup U_\beta$ and an element a such that $a \in (M'_i \cap \bigcup_{j\in\bigcup U_\beta} M^{\beta}_j)\setminus M^{\beta}_i$. There exists $\gamma<\beta$ such that $i \in \bigcup U_\gamma$ and there exists $j \in \bigcup U_\gamma$ such that $a \in M^{\gamma}_j$. Now consider the tower in ${}^{+}K^{*}_{\mu,U_\gamma}$, $(\bar{M}',\bar{a},\bar{N})\restriction U_\gamma$. Notice that $(\bar{M}',\bar{a},\bar{N})\restriction U_\gamma$ witnesses that $(\bar{M},\bar{a},\bar{N})^{\gamma}$ is not reduced. $\dashv$

This is where we need to introduce the additional assumption. It will allow us to take $<^{c}$-extensions of all scattered towers (not just the nice ones). For the remainder of the paper we assume that

Assumption 9.3. For every μ with $LS(K) < \mu < \kappa$, K^{am}_{μ} is closed under unions of length $<\mu^{+}$. In other words, for every μ with $LS(K) < \mu < \kappa$, and $\alpha < \mu^{+}$, if $\langle M_i \in K^{am}_{\mu} \mid i < \alpha\rangle$ is a $\prec_K$ increasing and continuous chain of models, then $\bigcup_{i<\alpha} M_i$ is an amalgamation base.

The proofs of the following two results on reduced towers (Proposition 9.4 and Theorem 9.5) rely on the $<^{c}$-extension property. The proofs were outlined in [ShVi] for the particular case of $U = \{[0,\alpha)\}$ when α is a limit ordinal $<\mu^{+}$ (see Fact 3.1.13 and Theorem 3.1.15 of their work). However, Shelah and Villaveces were unable to prove the $<^{c}$-extension property. The proofs of Proposition 9.4 and Theorem 9.5 for scattered towers are the same as for ${}^{+}K^{*}_{\mu,\alpha}$-towers.

Proposition 9.4 (Density of reduced towers). Let $(\bar{M},\bar{a},\bar{N}) \in {}^{+}K^{*}_{\mu,U}$ be given. Then there exists $(\bar{M}',\bar{a},\bar{N}) \in {}^{+}K^{*}_{\mu,U}$ such that

• $(\bar M, \bar a, \bar N) <^c (\bar M', \bar a, \bar N)$ and
• $(\bar M', \bar a, \bar N)$ is reduced.

Theorem 9.5 (Reduced towers are continuous). If $(\bar M, \bar a, \bar N) \in {}^+K^*_{\mu,U}$ is reduced, then $\bar M$ is continuous.

Definition 9.6 (Definition 3.2.1 of [ShVi]). For M a (μ,θ)-limit model,

(1) Let
$$St(M) := \left\{ (p, N) \;\middle|\; \begin{array}{l} N \prec_K M; \\ N \text{ is a } (\mu,\theta)-\text{limit model}; \\ M \text{ is universal over } N \text{ and} \\ p \in \text{ga-S}(M) \text{ does not } \mu-\text{split over } N \end{array} \right\},$$
and

(2) For types $(p_l, N_l) \in St(M)$ $(l = 1,2)$, we say $(p_1, N_1) \sim (p_2, N_2)$ iff for every $M' \in K^{am}_\mu$ extending M there is a $q \in S(M')$ extending both p_1 and p_2 such that q does not μ-split over N_1 and q does not μ-split over N_2.

Notice that $\sim$ is an equivalence relation on $St(M)$.

By Fact 2.19, we have

Fact 9.7. For $M \in K^{am}_\mu$, $|St(M)/\sim| \leq \mu$.

We can then consider towers which are saturated with respect to $St(M)$:

Definition 9.8. A tower $(\bar M, \bar a, \bar N) \in {}^+K^*_{\mu,U}$ is said to be full iff

(1) μ divides $\mathrm{cf}(\sup\{\bigcup U\})$ if μ is regular, otherwise, μ^ω divides $\mathrm{cf}(\sup\{\bigcup U\})$ and

(2) if $\beta \in \bigcup U$ and $(p, N^*) \in St(M_\beta)$, then for some $i < \mu$ with $\beta + i \in \bigcup U$, we have that $(\text{ga-tp}(a_{\beta+i}/M_{\beta+i}, M_{\beta+i+1}), N_{\beta+i}) \sim (p, N^*)$,

by μ dividing α we mean there exists θ such that $\alpha = \theta \cdot \mu$, where $\cdot$ is ordinal multiplication.

Remark 9.9. (1) Definition 9.8 appears in [ShVi] for the special case when $U = \{[0,\alpha)\}$ for α a limit ordinal $< \mu^+$ (see Definition 3.2.3 of their paper).

(2) Condition (1) of Definition 9.8 is used in the proof of Theorem 9.10

The following theorem is proved in [ShVi] under the particular instance of $U = \{[0,\alpha)\}$ for α a limit ordinal $< \mu^+$ (Theorem 3.2.4 of their work). We require the more general result, but the proof is similar to Shelah and Villaveces' argument.

Theorem 9.10. If $(\bar M, \bar a, \bar N) \in {}^+K^*_{\mu,U}$ is full and $\bar M$ is continuous, then $\bigcup_{i \in \bigcup U} M_i$ is a $(\mu, \mathrm{cf}(\sup\{\bigcup U\}))$-limit model over M_0.

In addition to these generalized results we need the following new theorem which is an analog to the statement that the union of $\kappa(T)$-many saturated models is saturated in first order stable theories. We are not implying that fullness is equivalent to saturation, but that the spirit of the result is similar. The following theorem was not stated in [ShVi] and is new:

Theorem 9.11 (Union of Full Towers is Full.) *Let α be a limit ordinal $< \mu^+$ and let $\mathcal{U}$ be set of intervals such that $|\mathcal{U}| < \mu^+$ and if μ is regular, μ divides $\mathrm{cf}(|\mathcal{U}|)$ otherwise, μ^{ω} divides $\mathrm{cf}(|\mathcal{U}|)$. If $\langle(\bar{M}^\beta, \bar{a}, \bar{N}) \in {}^{+}K^{*}_{\mu,\mathcal{U}} \mid \beta < \alpha\rangle$ is a $<^c$-increasing chain of full towers for $\alpha < \mu^+$, then the union is a full tower.*

Proof. Let $\langle(\bar{M}^\beta, \bar{a}, \bar{N}) \in {}^{+}K^{*}_{\mu,\mathcal{U}} \mid \beta < \alpha\rangle$ be a $<^c$-increasing chain of towers. We need to verify that for $i \in \mathcal{U}$ and $(p, N) \in \mathrm{St}(\bigcup_{\beta<\alpha} M_i^\beta)$, that there exists $j < \mu$ such that $i + j \in \mathcal{U}$ and $(p, N) \leq (\text{ga-tp}(a_{i+j}, \bigcup_{\beta<\alpha} M^\beta_{i+j}), N_{i+j})$.

By the definition of $<^c$, we have that $\bigcup_{\beta<\alpha} M_i^\beta$ is a (μ, α)-limit witnessed by $\langle M_i^\beta \mid \beta < \alpha\rangle$. By Theorem 7.2, there exists $\beta < \alpha$ such that p does not μ-split over M_i^β. Thus $(p \restriction M_i^{\beta+1}, M_i^\beta) \in \mathrm{St}(M_i^{\beta+1})$. By the assumption of fullness of the $\beta + 1^{st}$ tower, there exists a $j < \mu$ such that

$$(p \restriction M_i^{\beta+1}, M_i^\beta) \leq (\text{ga-tp}(a_{i+j}/M^{\beta+1}_{i+j}), N_{i+j}).$$

Recalling the definition of $\leq$, we know that there exists $q \in \text{ga-S}(\bigcup_{<\alpha} M_{i+j})$ such that

- $p \restriction M_i^\beta \subseteq q$
- $\text{ga-tp}(a_{i+j}/M^{\beta+1}_{i+j}) \subseteq q$
- q does not μ-split over M_i^β and
- q does not μ-split over N_{i+j}.

Notice that it suffices to show

Subclaim 9.12. $(p, N) \leq (\text{ga-tp}(a_{i+j}/\bigcup_{<\alpha} M_{i+j}), N_{i+j})$.

Proof of Subclaim 9.12. By definition of $\leq$, we have that

$$(p \restriction M_i^{\beta+1}, M_i^\beta) \leq (p, N).$$

Recalling that $\text{ga-tp}(a_{i+j}/\bigcup_{<\alpha} M_{i+j})$ does not μ-split over N_{i+j}, we see that

$$(\text{ga-tp}(a_{i+j}/M^{\beta+1}_{i+j}), N_{i+j}) \leq (\text{ga-tp}(a_{i+j}/\bigcup_{<\alpha} M_{i+j}), N_{i+j}).$$

Since $\leq$ is transitive, we have that $(p, N) \leq (\text{ga-tp}(a_{i+j}/\bigcup_{<\alpha} M_{i+j}), N_{i+j})$. $\dashv$

$\dashv$

10. Uniqueness of Limit Models

Recall the running assumptions:

(1) K is an abstract elementary class,
(2) K has no maximal models,
(3) K is categorical in some , > LS(K),
(4) GCH and $'_{„^+}(S''^+)$ holds for every cardinal „ < , and every regular with < „$^+$.
(5) For every „ with LS(K) < „ < , , $K^{am}_{„}$ is closed under unions of length „$^+$.

Under these assumptions, we can prove the uniqueness of limit models using the results from Sections 8 and 9. This is a partial solution to a conjecture from [ShVi].

Theorem 10.1 (Uniqueness of Limit Models). Let „ be a cardinal $_1$; $_2$ limit ordinal such that $_1$; $_2$ < „$^+$ • , . If M_1 and M_2 are („; $_1$) and („; $_2$) limit models over M , respectively, then there exists an isomorphism f : $M_1 \cong M_2$ such that f „M = id_M .

Proof. Let M 2 $K^{am}_{„}$ be given. By Proposition 2.29 wlog we may assume that $_1$ and $_2$ are regular. By Proposition 2.27 it su–ces to construct a model which is simultaneously a („; $_1$)-limit model and a („; $_2$)-limit model over M . Also by Proposition 2.27 we may assume that if „ is regular, „ divides $_2$ otherwise, ! divides $_2$. The idea is to build a (scattered) array of models such that at some point in the array, we will flnd a model which is a („; $_1$)-limit model witnessed by its height in the array and is a („; $_2$)-limit model witnessed by its horizontal position in the array, fullness and continuity. To guarantee that we have continuous towers, we will be constructing the array with reduced towers. We will deflne a chain of scattered towers of length „$^+$ £ $_1$ while increasing the index set of the towers as we proceed.

We will consider the index set $U^{(fi;\dagger)}$ at stage (fi;†) 2 „$^+$ £ $_1$ where

$$U^{(fi;\dagger)} := \bigcup f[fl„;fl„ \ _1 + „\dagger) \ j fl < fig:$$

Deflne by induction on (fi;†) 2 „$^+$ £ $_1$ the <^{c}-increasing and continuous sequence of scattered towers h(M′;a;N′)$^{(fi;\dagger)}$ 2$^+$ $K^{\prime}_{„;U^{(fi;\dagger)}}$ j (fi;†) 2 „$^+$ £ $_1$i, such that

(1) M `$_K$ $M_0^{(fi;\dagger)}$,
(2) (M′;a;N′)$^{(fi;\dagger)}$ is reduced,
(3) (M′;a;N′)$^{(fi;0)}$:= $\bigcup_{fl<fi}\bigcup_{\dagger<\ _1}$ (M′;a;N′)$^{(fl;\dagger)}$ and
(4) in successor stages new intervals of length „ put in representatives of all S t-types from the previous stages, more speciflcally, if there exists an interval u of length „ in $U^{fi;\dagger+1}$ n $U^{fi;\dagger}$, then for every i 2 $\bigcup U^{fi;\dagger}$ with i < minfug and every (p;N) 2 S t(M $_i$) there exists j 2 u such that (p;N) » (ga-tp(a_j=M $_j$);N $_j$).

This construction is possible:

$(0,0)$: We can choose $\langle M^*_i \mid i \in U^{(0,0)}\rangle$ to be an arbitrary $\prec_K$ increasing sequence of limit models of cardinality μ, with $M^*_0 = M$. For each $i \in U^{(0,0)}$ whenever $i+1 \in U^{(0,0)}$, fix $a^{(0,0)}_i \in M^*_{i+1} \setminus M^*_i$. Now consider ga-tp$(a^{(0,0)}_i / M^*_i)$. Since $M^{(0,0)}_i$ is a limit model, we can apply Theorem 7.2 to fix $N^{(0,0)}_i \in K^{am}_\mu$ such that ga-tp$(a^{(0,0)}_i/M^*_i)$ does not μ-split over $N^{(0,0)}_i$ and M^*_i is universal over $N^{(0,0)}_i$. By Theorem 9.4, there exists $\bar{M}^{(0,0)}$ such that $(\bar{M}^{(0,0)}, \bar{a}^{(0,0)}, \bar{N}^{(0,0)})$ is a member of $^{+}K^*_{\mu,U^{(0,0)}}$, is a $<^c$-extension of $(\bar{M}^*, \bar{a}^{(0,0)}, \bar{N}^{(0,0)})$ and is reduced.

$(\alpha,0)$: Take $(\bar{M},\bar{a},\bar{N})^{(\alpha,0)} := \bigcup_{\beta<\alpha}\bigcup_{\epsilon<\omega_1}(\bar{M},\bar{a},\bar{N})^{(\beta,\epsilon)}$

$(\alpha,\epsilon+1)$: Suppose that $(\bar{M},\bar{a},\bar{N})^{(\alpha,\epsilon)}$ has been defined. If $U^{(\alpha,\epsilon)}$ contains no new intervals of length μ, then by Theorem 9.4, we may take $(\bar{M},\bar{a},\bar{N})^{(\alpha,\epsilon+1)}$ to be a reduced $<^c$-extension of $(\bar{M},\bar{a},\bar{N})^{(\alpha,\epsilon)}$ in $^{+}K^*_{\mu,U^{(\alpha,\epsilon)}}$.

Suppose that $U^{(\alpha,\epsilon+1)}$ contains new intervals $\{u^\beta \mid \beta < \beta^0 < \mu\}$ each of length μ. Let $u^\beta = \{u^\beta_l \mid l < \mu\}$ be an enumeration of u^β. By Theorem 9.4 we can find a reduced extension $(\bar{M},\bar{a},\bar{N})^0 \in {}^{+}K^*_{\mu,U^{(\alpha,\epsilon+1)}\setminus\{u^\beta \mid \beta<\beta^0\}}$ of $(\bar{M},\bar{a},\bar{N})^{(\alpha,\epsilon)}$.

By Fact 9.7 we can enumerate $\bigcup_{j<\min u^\beta, j\in U^{(\alpha,\epsilon)}} \mathrm{St}(M_j)$ as $\{(p,N)^{u^\beta}_l \mid l<\mu\}$. By Corollary 8.8 and Theorem 9.4 we can find a reduced extension $(\bar{M},\bar{a},\bar{N})^{(\alpha,\epsilon+1)} \in {}^{+}K^*_{\mu,U^{(\alpha,\epsilon+1)}}$ of $(\bar{M},\bar{a},\bar{N})^0$ such that for every $l<\mu$ and $\beta<\beta^0$, $(p,N)^{u^\beta}_l \sim$ ga-tp$(a_{u^\beta_l}/M_{u^\beta_l}, N_{u^\beta_l})$. This completes the construction.

Consider the mapping $f : \mu^+ \to \mu^+$ defined by

$$f(\alpha) := \min\left\{\alpha' \;\middle|\; \begin{array}{l} \text{for every } \beta<\alpha,\ \epsilon<\omega_1,\ i\in\bigcup U^{(\beta,\epsilon)} \text{ and} \\ \text{for every } (p,N)\in \mathrm{St}(M^{(\beta,\epsilon)}_i) \text{ there} \\ \text{exists } \beta^0\le\alpha' \text{ and } j<\mu \text{ such that} \\ (\text{ga-tp}(a_{i+j}/M^{(\beta^0,0)}_{i+j}), N_{i+j}) \sim (p,N) \end{array}\right\}.$$

By condition (A) of the construction, f can be defined. Then there exists a club C such that

$$\delta\in C \Rightarrow f\restriction\delta : \delta\to\delta.$$

Notice that by the definition of f, this implies

$$(*)\quad \delta\in C\cap S^{\lambda^+}_{\aleph_0} \Rightarrow (\bar{M},\bar{a},\bar{N})^{(\delta,0)} \text{ is full.}$$

Pick $\alpha\in C\cap S^{\lambda^+}_{\aleph_0}$.

Subclaim 10.2. We can find $\langle\alpha_\epsilon<\mu^+ \mid \epsilon\le\omega_1\rangle$, an increasing and continuous sequence of ordinals, α, such that $(\bar{M},\bar{a},\bar{N})^{(\alpha_\epsilon,0)}\restriction U^{(\alpha,0)}$ is full.

Proof of Subclaim 10.2. Take $\langle\alpha_\epsilon<\mu^+ \mid \epsilon\le\omega_1\rangle$ to be an increasing and continuous sequence of ordinals α from C. By definition of f, $(\bar{M},\bar{a},\bar{N})^{(\alpha_\epsilon,0)}$ satisfies the second condition of the definition of full as follows: