DRAFT - Categoricity and Stability in Abstract Elementary Classes

by

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This thesis is dedicated to my daughter Ariella Ronit.

Dedication page

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CHAPTER I

Introduction

The purpose of this introduction is to describe the program of classification theory of non-elementary classes with respect to categoricity and stability. This thesis tackles the classification theory of non-elementary classes from two perspectives. In Chapter II we work towards a categoricity transfer theorem, while Chapter III focuses on the development of a stability theory for abstract elementary classes. At the end of this chapter we provide a brief outline of the thesis.

Early work in model theory was closely tied to other areas of mathematics. Led by Robinson, Malcev and Tarski, model theorists worked on generalizing known theorems about fields to arbitrary first order theories. In the sixties, James Ax and Simon Kochen found far reaching applications of model theory to the theory of valued fields. Their work on Hensel fields and p-adic numbers was used to refute a conjecture of Artin [CK]. Current work in model theory can be classified as either stemming from theorems and conjectures in algebra or motivated by pure model-theoretic questions which may someday shed light on open questions in algebra.

The origins of much of pure model theory can be traced back to Łŏs' Conjecture, one of the most influential conjectures in model theory, motivated by an algebraic result of Steinitz from 1915. Steinitz's Theorem states that for every uncountable cardinal, λ, there is exactly one algebraically closed field of characteristic p of cardinality λ (up to isomorphism). In 1954, Łŏs conjectured that elementary classes mimic the behavior of algebraically closed fields:

Conjecture I.0.1. *If T is a countable first order theory and there exists a cardinal $\lambda > \aleph_0$ such that T has exactly one model of cardinality λ (up to isomorphism), then for every $\mu > \aleph_0$, T has exactly one model of cardinality μ.*

This conjecture was resolved by Michael Morley in his Ph.D. thesis in 1962 [Mo]. Morley then questioned the status of the conjecture for uncountable theories. Building on work of W. Marsh, F. Rowbottom and J.P. Ressayr, S. Shelah proved the statement for uncountable theories in 1970 [Sh31].

The theorem which affirmitavely resolves Łŏs' Conjecture is often referred to as Morley's Categorcity Theorem, which motivates the following terminology:

Definition I.0.2. A theory T is said to be *categorical in* λ if and only if there is exactly one model of T of cardinality λ up to isomorphism.

Out of Morley and Shelah's proofs, fundamental techniques and concepts such as prime models, rank functions, superstable theories, stable theories and minimal types surfaced. Present day research in first order model theory, particularly *stability theory* or *classification theory*, would be unrecognizable without these techniques and concepts. Model theorists have used the techniques and concepts of stability theory to answer open questions in algebraic geometry.

While first order logic has far reaching applications in other fields of mathematics, there are several interesting frameworks which cannot be captured by first order logic. For example, non-archimedian fields, Noetherian rings, locally finite groups and finite structures cannot be axiomatized by first order logic. Building on the work of Erdos-

Tarski, Hanf, D. Scott, Lopez-Escobar and C. Karp, model theorists C.C. Chang and H.J. Kiesler made much progress in the study of non-first order logics including $L(\mathbf{Q})$ and $L_{\omega_1,\omega}$ [CK],[Ke1], [Ke2]. $L(\mathbf{Q})$ is an extension of first order logic with the addition of a quantifier $\mathbf{Q}$, where $\mathbf{Q}$ is interpretted as *there exists at least* $\aleph_1$. $L_{\omega_1,\omega}$ is also an extension of first order logic allowing for countable disjunctions and conjunctions.

A major breakthrough in non-first-order model theory occured in 1974 when Shelah answered John Baldwin's question (which was made in the early 1970s and reproduced on Harvey Friedman's list of open problems):

Problem I.0.3. Does there exists a countable similarity type and a countable $T \subseteq L(\mathbf{Q})$ (in the $\aleph_1$ interpretation) such that T has a unique uncountable model (up to isomorphism)?

Shelah's solution to this problem in the mid-seventies indicated a strong link between categorical theories and the existence of models in uncountable cardinals [Sh 48]. The solution prompted Shelah to pose a generalization of Łŏs' Conjecture to $L_{\omega_1,\omega}$ as a test question to measure progress in non-first-order model theory.

Conjecture I.0.4. *If φ is an $L_{\omega_1,\omega}$ theory categorical in some $\lambda > Hanf(\varphi)$ then φ is categorical in every $\mu > Hanf(\varphi)$.*

In the late seventies Shelah identified the notion of *abstract elementary class (AEC)* to capture many non-first-order logics [Sh 88] including $L_{\omega_1,\omega}(\mathbf{Q})$. An abstract elementary class is a class of structures of the same similarity type endowed with a morphism satisfying natural properties such as closure under directed limits.

Definition I.0.5. $\mathcal{K}$ is an *abstract elementary class (AEC)* iff $\mathcal{K}$ is a class of models for some vocabulary τ and is equipped with a binary relation, $\preceq_{\mathcal{K}}$ satisfying the following:

(1) Closure under isomorphisms.

(2) $\preceq_{\mathcal{K}}$ refines the submodel relation.

(3) $\preceq_{\mathcal{K}}$ is a partial order on $\mathcal{K}$.

(4) If $\langle M_i \mid i < \delta \rangle$ is a $\prec_{\mathcal{K}}$-increasing and chain of models in $\mathcal{K}$

   (a) $\bigcup_{i<\delta} M_i \in \mathcal{K}$,

   (b) for every $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$ and

   (c) if $M_i \prec_{\mathcal{K}} N$ for every $i < \delta$, then $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} N$.

(5) If $M_0, M_1 \preceq_{\mathcal{K}} N$ and M_0 is a submodel of M_1, then $M_0 \preceq_{\mathcal{K}} M_1$.

(6) (Downard Löwenheim-Skolem Axiom) There is a Löwenheim-Skolem number of $\mathcal{K}$, denoted $LS(\mathcal{K})$ which is the minimal κ such that for every $N \in \mathcal{K}$ and every $A \subset N$, there exists M with $A \subseteq M \prec_{\mathcal{K}} N$ of cardinality $\kappa + |A|$.

This has led Shelah to restate his conjecture in the following form:

Conjecture I.0.6 (Shelah's Categoricity Conjecture). *Let $\mathcal{K}$ be an abstract elementary class. If $\mathcal{K}$ is categorical in some $\lambda > Hanf(\mathcal{K})$, then for every $\mu > Hanf(\mathcal{K})$, $\mathcal{K}$ is categorical in μ.*

Despite the existence of over 500 published pages of partial results towards this conjecture, it remains very open. Similar to the solution to Łŏs' conjecture, a solution of Shelah's categoricity conjecture is expected to provide the basic conceptual tools necessary for a stability theory for non-first order logic. This enhances the potential for further applications of model theory to other areas of mathematics.

Since the mid-eighties, model theorists have approached Shelah's conjecture from two different directions. Shelah, M. Makkai and O. Kolman attacked the conjecture with set theoretic assumptions [MaSh], [KoSh], [Sh 472]. On the other hand, Shelah also looked at the conjecture under additional model theoretic assumptions [Sh 394], [Sh 600]. More recent work of Shelah and A. Villaveces [ShVi] profits from both model theoretic and set theoretic assumptions, however these assumptions are weaker than the hypothesis made in [MaSh], [KoSh], [Sh 472], [Sh 394], and [Sh 600]. Shelah and Villaveces make the following assumptions:

Assumption I.0.7. *(1) $\mathcal{K}$ is an AEC with no maximal models with respect to the relation $\prec_{\mathcal{K}}$,*

(2) GCH holds and

(3) a form of the weak diamond holds, namely $\Phi_{\mu^+}(S^{\mu^+}_\theta)$ holds for every regular θ with $\theta \leq \mu$.

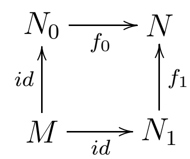
A central emphasis of Chapter II is to resolve problems from [ShVi] and to work towards a solution to Shelah's conjecture in this framework.

Let us recall some definitions in AECs which differ from the first-order counterparts. Because of the category-theoretic definition of abstract elementary classes, the first order notion of formulas and types cannot be applied. To overcome this barrier, Shelah has suggested identifying types, not with formulas, but with the orbit of an element under the group of automorphisms fixing a given structure. In order to carry out a sensible definition of type, the following binary relation E must be an equivalence relation on triples (a, M, N). In order to avoid confusing this new notion of "type" with the conventional one (i.e. set of formulas) we will follow [Gr1] and [Gr2] and introduce it below under the name of *Galois type*.

Definition I.0.8. For triples $(\bar{a}_l, M_l, N_l)$ where $\bar{a}_l \in N_l$, $M_l, N_l \in \mathcal{K}$ for $l = 0, 1$, we define a binary relation E as follows:

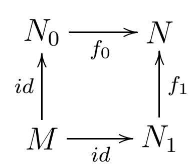
$$(\bar{a}_0, M_0, N_0)E(\bar{a}_1, M_1, N_1) \text{ iff}$$

$M := M_0 = M_1$ and there exists $N \in \mathcal{K}$ and $\prec_{\mathcal{K}}$-mappings f_0, f_1 such that for $l = 0, 1$ $f_l : N_l \to N$, $f_l \restriction M = id_M$ and $f_0(\bar{a}_0) = f_1(\bar{a}_1)$.



To prove that E is an equivalence relation (more specifically, that E is transitive), we need to restrict ourselves to amalgamation bases.

Definition I.0.9. Let $\mathcal{K}$ be an AEC. A model $M \in \mathcal{K}$ is said to be an (μ_0, μ_1)-*amalgamation base* if and only if for every $N_i \in \mathcal{K}$ of cardinality μ_i with $M \prec_{\mathcal{K}} N_i$ for $i = 0, 1$, there exists a model $N \in \mathcal{K}$ and $\prec_{\mathcal{K}}$-mappings $f_0 : N_0 \to N$ and $f_1 : N_1 \to N$ such that the following diagram commutes:



When $\mu_0 = \mu_1 = \|M\|$, we say that M is an *amalgamation base*.

We can now define types in terms of this equivalence relation:

Definition I.0.10. For $M, N \in \mathcal{K}$ with M, N amalgamation bases and $\bar{a}$, a finite sequence in N, the *(Galois-)type of $\bar{a}$ in N over M*, written ga-tp$(\bar{a}/M, N)$, is defined to be $(\bar{a}, M, N)/E$.

Remark I.0.11. Unlike the first-order definition of type, this definition depends on not only M and N, but also the class $\mathcal{K}$. Subtlities such as this commonly arise when generalizing first-order notions to the context of AECs. With this in mind, consequences which may seem trivial in the first order context, will have far deeper proofs in the context of AECs.

In 1985 Rami Grossberg made the following conjecture:

Conjecture I.0.12. *If $\mathcal{K}$ is a categorical AEC, then every $M \in \mathcal{K}$ is an amalgamation base.*

This conjecture encouraged Shelah to produce a partial solution to the categoricity conjecture under the assumption that every model $M \in \mathcal{K}$ is an amalgamation base [Sh 394]. This result directs future work towards the categoricity conjecture to solving Conjecture I.0.12. The underlying goal of [ShVi] was to make progress towards Conjecture I.0.12 under Assumptions II.1.1. Not knowing that every model is an amalgamation base presents several obstacles in applying known notions and techniques. For instance, there may exist some models over which we cannot even define the most basic notion of a type. New approaches have been identified and explored in [ShVi] and in Chapter II of this thesis.

One approach to Conjecture I.0.12 is to see if arguments from [KoSh] can be carried out in this more general context. Shelah and Kolman prove Conjecture I.0.12 for $L_{\kappa,\omega}$ theories where κ is a measurable cardinal. They first introduce limit models as a substitute for saturated models, and then prove the uniqueness of limit models. A major objective of [ShVi] was to show the uniqueness of limit models.

In the Fall of 1999, I identified a gap in Shelah and Villaveces' proof of uniqueness of limit models. As of the Fall of 2001, Shelah and Villaveces could not resolve the problem. The goal of Chapter II is to prove the uniqueness of limit models.

The main attraction to solving Shelah's Conjecture is to harvest the proof in order to develop stability theory for abstract elementary classes. It is with the stability theory in first order logic that model theoretic proofs are applied to other mathematical fields. Thus having a stability theory for abstract elementary classes provides the potential for further applications of model theory to other areas.

By investigating work towards Shelah's Conjecture, one may eliminate the assumption of categoricity and develop a stability theory. The notion of splitting that appears in [Sh 394] can be studied in stable AECs. Rami Grossberg and I identified a nicely behaved, yet general class of AECs (*tame AECs* see Definition III.4.2) in which non-splitting can be exploited. We begin developing a stability theory by proving the existence of Morley Sequences in tame, stable AECs. This is the subject of Chapter III.

The structure of the remainder of the thesis follows. Each chapter begins with a brief introduction and an outline of the chapter.

Chapter II We solve a conjecture of [ShVi] by proving the uniqueness of limit models in a categorical AEC with no maximal models under some mild set theoretic assumptions. The uniqueness of limit models suggests that limit models are the right substitute for saturation when considering Shelah's Categoricity Conjecture. In this chapter, we provide an exposition of results from [ShVi] featuring a proof that limit models are amalgamation bases using a version of Devlin-Shelah's weak diamond. We introduce the notion of nice towers to resolve a problem from [ShVi] in proving the extension property for towers. In order to prove the uniqueness of limit models, we prove the extension property for non-splitting types. This result does not rely on categoricity and will be used in

Chapter III to prove the existence of Morley sequences. This chapter includes two other new theorems: the union of full towers is full and reduced towers are continuous.

Chapter III Some background on AECs required for this chapter is included in Section 2.2 of Chapter II. Chapter III focuses on developing a stability theory for AECs. We introduce a nicely behaved class of AECs, tame AECs, in which consistency has small character. Showing that a categorical AEC is tame is a common step in partial solutions to Shelah's Categoricity Conjecture. In this chapter, we prove the existence of Morley Sequences for tame, stable AECs. Up until this point the only known proofs of existence of indiscernible sequences in general AECs has been under the assumption of categoricity using Ehrenfreucht-Mostowski models. Our proof does not use categoricity. The existence of Morley sequences suggests a notion of dividing which may be used to prove a stability spectrum theorem for tame AECs.

CHAPTER II

Towards a Categoricity Theorem for Abstract Elementary Classes

2.1 Introduction

Shelah's paper, [Sh 702] is based on a series of lectures given at Rutgers University. In the lectures, Shelah elaborates on open problems in model theory which he has attempted but which have not yet been solved. There Shelah refers to the subject of Section 13, "Classification of Non-elementary Classes," as the major problem of model theory. He points out that one of the main steps in classifying non-elementary classes is the development of stability theory. In first order logic, solutions to Łŏs' Conjecture produced machinery that advanced the study of stability theory. It is natural, then, to consider a generalization of this conjecture as a test question for a proposed stability theory for AECs (Conjecture I.0.6)

Despite the existence of over 500 published pages of partial results towards this conjecture, it remains very open. Since the mid-eighties, model theorists have approached Shelah's conjecture from two different directions. Shelah, M. Makkai and O. Kolman attacked the conjecture with set theoretic assumptions (see [MaSh], [KoSh] and [Sh 472]). On the other hand, Shelah also looked at the conjecture under additional model theoretic assumptions in [Sh 394] and [Sh 600]. More recent work of

Shelah and A. Villaveces [ShVi] profits from both model theoretic and set theoretic assumptions, however these assumptions are weaker than the hypotheses made in [MaSh], [KoSh], [Sh 472], [Sh 394], and [Sh 600]. A main feature of their context is that they work in AECs where the amalgamation property is not known to hold. This chapter focuses on resolving problems from [ShVi]. Here we recall the context of [ShVi] (AssumptionsII.1.1.(1) through II.1.1.(5)).

Assumption II.1.1. *We make the following assumptions for the remainder of this chapter:*

(1) $\mathcal{K}$ is an abstract elementary class,

(2) $\mathcal{K}$ has no maximal models,

(3) $\mathcal{K}$ is categorical in some $\lambda > LS(\mathcal{K})$,

(4) GCH holds and

(5) $\Phi_{\mu^+}(S^{\mu^+}_\theta)$ holds for every cardinal $\mu < \lambda$ and every regular θ with $\theta < \mu^+$.

Assumption II.1.1.(5) is not explicitly made in [ShVi]. We believe this version of weak diamond is needed to carry out Shelah and Villaveces' suggestion for the proof that limit models are amalgamation bases. We provide a complete proof of the theorem which uses Assumption II.1.1.(5) (see Theorem II.4.3) and give an exposition of the strength of Assumption II.1.1.5 in Section 2.4.

In light of Conjecture I.0.12 and the downward solution to Conjecture I.0.6 under the assumption of the amalgamation property, work towards Conjecture I.0.6 is directed towards deriving the amalgamation property from categoricity. The underlying goal of [ShVi] was to make progress towards Conjecture I.0.12 under Assumption II.1.1. Not knowing that every model is an amalgamation base presents several obstacles in applying known notions and techniques. For instance, there may exist some models over which we cannot even define the most basic notion of a type.

One approach to Conjecture I.0.12 is to see if arguments from [KoSh] can be carried out in this more general context. Shelah and Kolman prove Conjecture I.0.12 for $L_{\kappa,\omega}$ theories where κ is a measurable cardinal. They first introduce limit models as a substitute for saturated models, and then prove the uniqueness of limit models. A major objective of [ShVi] was to show the uniqueness of limit models:

Conjecture II.1.2 (Uniqueness of Limit Models). *Suppose Assumption II.1.1 holds. For $\theta_1, \theta_2 < \mu^+ < \lambda$, if M_1 and M_2 and (μ, θ_1)-, (μ, θ_2)-limit models over M, respectively, then M_1 is isomorphic to M_2.*

While limit models were used to prove that every model is an amalgamation base in [KoSh], limit models played a *behind-the-scenes* role in Shelah's downward solution to the categoricity conjecture in [Sh 394]. Furthermore, there is evidence that the uniqueness of limit models provides a basis for the development of a notion of non-forking and a stability theory for abstract elementary classes. Limit models are used in Chapter III to produce Morley sequences in tame and stable AECs. They also appear in [Sh 600] as an axiom for frames.

In all of these applications, limit models provide a substitute for saturation. Without the amalgamation property, it is unknown how to prove the uniqueness of saturated models. This may seem strange, because the proof is so straight-forward in the first order case. However, since we only have types over amalgamation bases (not arbitrary sets), the usual back-n-forth argument cannot be carried out. Even with the amalgamation property, the back-n-forth consrtuction is non-trivial (see [Gr] for details). Since we are working in a context without the luxury of the amalgamation property, in order for limit models to provide a reasonable substitute for saturated models, there must be a uniqueness theorem. This is the main result of this chapter.

Here we outline the structure of this chapter:

Section 2.1 We connect the uniqueness of limit models with its role in understanding Shelah's Categoricity Conjecture for AECs, the amalagamation property and stability theory for AECs. An outline of the remainder of the chapter is given.

Section 2.2 In this section we provide some of the necessary definitions for AECs including the amalgamation property and limit models. This background is also used in Chapter III.

Section 2.3 We provide a description of an index set used to prove the existence of universal models and to prove weak disjoint amalgamation. We summarize a few properties of EM reducts constructed with this index set. Because of categoricity, we can view every model of $\mathcal{K}$ as a $\mathcal{K}$-substructure of an EM reduct.

Section 2.4 Using a version of the weak diamond, we provide a complete proof of a fact from [ShVi] that limit models are amalgamation bases. This allows us to show the existence of limit models.

Section 2.5 We provide a complete proof of Shelah and Villaveces' Weak Disjoint Amalgamation Theorem. This theorem will be used in constructing extensions of towers. The proof uses the EM models which were described in Section 2.3.

Section 2.6 In the next few sections we will be introducing classes of towers. Ultimately, we will only use scattered towers to prove the uniqueness of limit models. However, to make the proof of the extension property for scattered towers more manageable, we begin with naked towers and slowly modify them.

We will show that for every tower $(\bar{M}, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ can properly extended (with respect to the ordering $<^b_{\mu,\alpha}$) to a larger tower in $\mathcal{K}^*_{\mu,\alpha}$. This closes one of the gaps from [ShVi]. The proof utilizes directed systems and direct limits. The reader is suggested to refer to Section 2.2 for a discussion of these concepts in AECs.

Section 2.7 We define the notion of splitting for AECs and prove the extension property for non-splitting. This result does not rely on the categoricity assumption. We will use the extension property for non-splitting in Chapter III as well. We also recall Shelah and Villaveces' result concerning splitting chains (Theorem II.7.2). After analyzing their proof we are able to read out a very useful corollary which serves as a substitute for $\kappa(T)$ for non-splitting. We then augment the towers from Section 2.6 with non-splitting types. We prove the extension property for this class of towers as well. The proof relies on understanding the $<^b_{\mu,\alpha}$-extension property from Section 2.6.

Section 2.8 We begin this section with a description of the structure of the proof of the uniqueness of limit models. We now make the final modification for towers by adjusting the index set from an ordinal to a collection of intervals of ordinals and prove an extension property for this class. This is a new theorem. The proof relies on the proofs from Section 2.6 and Section 2.7 and on the results about non-splitting.

Section 2.9 One of the problems with our chains of towers is that $<^c$-extensions are often discontinuous. We provide a complete proof that reduced towers are continuous. This solves another problem from [ShVi]. The proof relies on the non-splitting results from Section 2.7. We then conclude that every scattered tower has a continuous $<^c$-extension.

Section 2.10 In this section we introduce full towers which are towers which realize many stationarizations of types. We then show that the top of a full continuous tower is a limit model. We also prove a new result, that the union of full towers is full.

Section 2.11 Here we prove Conjecture II.1.2. The proof uses the extension property for scattered towers and the results on reduced and full towers.

2.2 Background

Recall the definition of an abstract elementary class from the introduction (Definition I.0.5.)

Notation II.2.1. If λ is a cardinal and $\mathcal{K}$ is an abstract elementary class, $\mathcal{K}_\lambda$ is the collection of elements of $\mathcal{K}$ with cardinality λ.

Definition II.2.2. For models M, N in an AEC, $\mathcal{K}$, the mapping $f : M \to N$ is an $\prec_{\mathcal{K}}$-*embedding* iff f is an injective $L(\mathcal{K})$-homomorphism and $f[M] \preceq_{\mathcal{K}} N$.

Using the axioms of AEC, one can show that Axiom 4 has an alternative formulation (see [Sh 88] or Chapter 13 of [Gr]):

Definition II.2.3. A partially ordered set $(I, \leq)$ is *directed* iff for every $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$.

Proposition II.2.4 (P.M. Cohn 1965). *Let* $(I, \leq)$ *be a directed set. If* $\langle M_t \mid t \in I\rangle$ *and* $\{h_{t,r} \mid t \leq r \in I\}$ *are such that*

(1) for $t \in I$, $M_t \in \mathcal{K}$

(2) for $t \leq r \in I$, $h_{t,r} : M_t \to M_r$ *is a* $\prec_{\mathcal{K}}$*-embedding and*

(3) for $t_1 \leq t_2 \leq t_3 \in I$, $h_{t_1,t_3} = h_{t_2,t_3} \circ h_{t_1,t_2}$ *and* $h_{t,t} = id_{M_t}$,

then, whenever $s = \lim_{t \in I} t$, *there exist* $M_s \in \mathcal{K}$ *and* $\prec_{\mathcal{K}}$*-mappings* $\{h_{t,s} \mid t \in I\}$ *such that*

$$h_{t,s} : M_t \to M_s, M_s = \bigcup_{t<s} h_{t,s}(M_t) \text{ and}$$
$$\text{for } t_1 \leq t_2 \leq s, h_{t_1,s} = h_{t_2,s} \circ h_{t_1,t_2} \text{ and } h_{s,s} = id_{M_s}.$$

Definition II.2.5. (1) $(\langle M_t \mid t \in I\rangle, \{h_{t,s} \mid t \leq s \in I\})$ from Proposition II.2.4 is called a *directed system*.

(2) We say that M_s together with $\langle h_{t,s} \mid t \leq s\rangle$ satisfying the conclusion of Proposition II.2.4 is *a direct limit of* $(\langle M_t \mid t < s\rangle, \{h_{t,r} \mid t \leq r < s\})$.

In fact we can conclude more about direct limits (Lemma II.2.6). We will use this lemma in our proofs of the extension property for towers.

Lemma II.2.6. *Suppose that* $\langle M_t \prec_{\mathcal{K}} N_t \mid t \in I\rangle$ *and* $\langle f_{t,s} \mid t \leq s \in I\rangle$ *is a directed system with* $f_{t,s} : N_t \to N_s$ *and* $f_{t,s} \restriction M_t : M_t \to M_s$. *Then we can find a direct limit* $(N^*, \langle f_{t,\sup\{I\}} \mid t \in I\rangle)$ *of* $(\langle N_t \mid t \in I\rangle, \langle f_{t,s} \mid t \leq s \in I\rangle)$ *and* $(M^*, \langle g_{t,\sup\{I\}} \mid t \in I\rangle)$ *a direct limit of* $(\langle M_t \mid t \in I\rangle, \langle f_{t,s} \restriction M_t \mid t \leq s \in I\rangle)$ *such that* $M^* \prec_{\mathcal{K}} N^*$ *and* $f_{t,\sup\{I\}} \restriction M_t = g_{t,\sup\{I\}}$.

The proof of Lemma II.2.6 is straight-forward using the following proposition:

Proposition II.2.7 ([Sh 88] or see [Gr]). $\mathcal{K}^{\prec_{\mathcal{K}}} := \{(N, M) \mid M, N \in \mathcal{K}, M \prec_{\mathcal{K}} N\}$ *is an abstract elementary class with* $L(\mathcal{K}^{\prec_{\mathcal{K}}}) = L(\mathcal{K}) \bigcup \{P\}$ *where* P *is a unary predicate and* $\prec_{\mathcal{K}^{\prec_{\mathcal{K}}}}$ *is defined by*

$$(N, M) \prec_{\mathcal{K}^{\prec_{\mathcal{K}}}} (N', M') \Leftrightarrow (N \prec_{\mathcal{K}} N' \text{ and } M \prec_{\mathcal{K}} M').$$

We will use Lemma II.2.6 as well as the trivial observation (Claim II.2.8) in the proof of the Conjecture II.1.2.

Claim II.2.8. *If $\langle N_t \mid t < s\rangle$ and $\langle f_{r,t} \mid r < t < s\rangle$ form a directed system and for every $r \leq t < s$ we have that $N_t = N_r = N$ and $f_{r,t} \in Aut(N)$. Then a direct limit $(N_s, \langle f_{t,s} \mid t \leq s\rangle)$ of this system is such that $f_{t,s} : N_t \cong N_s$ for every $t \leq s$. Moreover we can choose a direct limit such that $N_s = N$.*

The following gives a characterization of AECs as PC-classes. Theorem II.2.10 is often referred to as Shelah's Presentation Theorem.

Definition II.2.9. A class $\mathcal{K}$ of structures is called a $PC-class$ if there exists a language L_1, a first order theory, T_1, in the language, L_1, and a collection of types without parameters, Γ, such that L_1 is an expansion of $L(\mathcal{K})$ and

$$\mathcal{K} = PC(T_1, \Gamma, L) := \{M \restriction L : M \models T_1 \text{ and } M \text{ omits all types from } \Gamma\}.$$

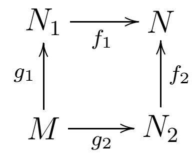
When $|T_1| + |L_1| + |\Gamma| + \aleph_0 = \mu$, we say that $\mathcal{K}$ is PC_μ.

Theorem II.2.10 (Lemma 1.8 of [Sh 88] or [Gr]). *If $(\mathcal{K}, \prec_{\mathcal{K}})$ is an AEC, then there exists $\mu \leq 2^{LS(\mathcal{K})}$ such that $\mathcal{K}$ is PC_μ.*

In Section 2.3 we will see that this presentation of AECs as PC-classes allows us to construct Ehrenfuecht-Mostowski models.

Definition II.2.11. Let $\mathcal{K}$ be an abstract elementary class.

(1) Let μ,κ_1 and κ_2 be cardinals with $\mu \leq \kappa_1, \kappa_2$. We say that $M \in \mathcal{K}_\mu$ is a (κ_1, κ_2)-*amalgamation base* if for every $N_1 \in \mathcal{K}_{\kappa_1}$ and $N_2 \in \mathcal{K}_{\kappa_2}$ and $g_i : M \to N_i$ for $(i = 1, 2)$, there are $\prec_{\mathcal{K}}$-embeddings f_i, $(i = 1, 2)$ and a model N such that the following diagram commutes:



(2) We say that a model $M \in \mathcal{K}_\mu$ is an *amalgamation base* if M is a (μ, μ)-amalgamation base.

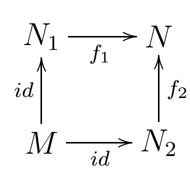
(3) We write $\mathcal{K}^{am}$ for the class of amalgamation bases which are in $\mathcal{K}$.

(4) We say $\mathcal{K}$ satisfies the *amalgamation property* iff for every $M \in \mathcal{K}$, M is an amalgamation base.

Remark II.2.12. We get an equivalent definition of amalgamation base, if we additionally require that $g_i \upharpoonright M = id_M$ for $i = 1, 2$, in the definition above. See [Gr] for details.

Amalgamation bases are central in the definition of types. Since we are not working in a fixed logic, we will not define types as collections of formulas. Instead, we will define types as equivalence classes with respect to images under $\prec_{\mathcal{K}}$-mappings:

Definition II.2.13. For triples $(\bar{a}_l, M_l, N_l)$ where $\bar{a}_l \in N_l$ and $M_l \preceq_{\mathcal{K}} N_l \in \mathcal{K}$ for $l = 0, 1$, we define a binary relation E as follows: $(\bar{a}_0, M_0, N_0) E (\bar{a}_1, M_1, N_1)$ iff $M_0 = M_1$ and there exists $N \in \mathcal{K}$ and $\prec_{\mathcal{K}}$-mappings f_0, f_1 such that $f_l : N_l \to N$ and $f_l \upharpoonright M = id_M$ for $l = 0, 1$ and $f_0(\bar{a}_0) = f_1(\bar{a}_1)$:



Remark II.2.14. E is an equivalence relation on the set of triples of the form $(\bar{a}, M, N)$ where $M \preceq_{\mathcal{K}} N$, $\bar{a} \in N$ and $M, N \in \mathcal{K}^{am}_{\mu}$ for fixed $\mu \geq LS(\mathcal{K})$.

In AECs with the amalgamation property, we are often limited to speak of types only over models. Here we are further restricted to deal with types only over models which are amalgamation bases.

Definition II.2.15. Let $\mu \geq LS(\mathcal{K})$ be given.

(1) For $M, N \in \mathcal{K}^{am}_{\mu}$ and $\bar{a} \in {}^{\omega>}\mathrm{N}$, the *Galois-type of* $\bar{a}$ *in* N *over* M, written ga-tp$(\bar{a}/M, N)$, is defined to be $(\bar{a}, M, N)/E$.

(2) For $M \in \mathcal{K}^{am}_{\mu}$, ga-S$^1(M) := \{\text{ga-tp}(a/M, N) \mid M \preceq N \in \mathcal{K}^{am}_{\mu}, a \in N\}$.

(3) We say $p \in$ ga-S(M) is realized in N whenever $M \prec_{\mathcal{K}} N$ and there exist $a \in N$ and $N' \in \mathcal{K}^{am}_{\mu}$ such that $p = (a, M, N')/E$.

Remark II.2.16. We refer to these types as Galois-types to distinguish them from notions of types defined as a collection of formulas.

Proposition II.2.17 (see [Gr]). *When* $\mathcal{K} = \mathrm{Mod}(T)$ *for* T *a complete first order theory, the above definition of* ga-tp$(a/M, N)$ *coincides with the classical first order defintion where* c *and* a *have the same type over* M *iff for every first order formula* $\varphi(x, \bar{b})$ *with parameters from* M*,*

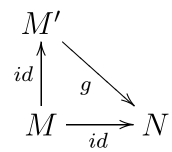
$$\models \varphi(c, \bar{b}) \leftrightarrow \models \varphi(a, \bar{b}).$$

Proof. By Robinson's Consistency Theorem. ⊣

Definition II.2.18. We say that $\mathcal{K}$ is *stable in* μ if for every $M \in \mathcal{K}^{am}_{\mu}$, $|\,\text{ga-S}^1(M)| = \mu$.

Fact II.2.19 (Fact 2.1.3 of [ShVi]). *Since* $\mathcal{K}$ *is categorical in* λ*, for every* $\mu < \lambda$*, we have that* $\mathcal{K}$ *is stable in* μ*.*

Definition II.2.20. (1) Let κ be a cardinal. We say N is *κ-universal over M* iff for every $M' \in \mathcal{K}_\kappa$ with $M \prec_{\mathcal{K}} M'$ there exists a $\prec_{\mathcal{K}}$-embedding $g : M' \to N$ such that $g \restriction M = id_M$:



(2) We say N is *universal over M* iff N is $\|M\|$-universal over M.

The existence of universal extensions follows from categoricity and GCH:

Lemma II.2.21 (Theorem 1.3.1 from [ShVi]). *For every μ with $LS(\mathcal{K}) < \mu < \lambda$, if $M \in \mathcal{K}^{am}_\mu$, then there exists $M' \in \mathcal{K}^{am}_\mu$ such that M' is universal over M.*

Notice that the following proposition asserts that it is unreasonable to prove a stronger existence statement than Lemma II.2.21, without having proved the amalgamation property.

Proposition II.2.22. *If M' is universal over M, then M is an amalgamation base.*

As mentioned in the introduction, limit models were introduced by Kolman and Shelah in [KoSh]. After proving the uniqueness of limit models in their context, Shelah and Kolman derive the Amalgamation Property. The main goal of this chapter is to prove the uniqueness of limit models in the context of [ShVi].

Definition II.2.23. For $M', M \in \mathcal{K}_\mu$ and σ a limit ordinal with $\sigma < \mu^+$, we say that M' is a *(μ, σ)-limit over M* iff there exists a $\prec_{\mathcal{K}}$-increasing and continuous sequence of models $\langle M_i \in \mathcal{K}_\mu \mid i < \sigma \rangle$ such that

(1) $M \preceq_{\mathcal{K}} M_0$,

(2) $M' = \bigcup_{i<\sigma} M_i$

(3) for $i < \sigma$, M_i is an amalgamation base and

(4) M_{i+1} is universal over M_i.

Remark II.2.24. (1) Notice that in Definition II.2.23, for $i < \sigma$ and i a limit ordinal, M_i is a (μ, i)-limit model.

(2) Notice that Condition (4) implies Condition (3) of Definition II.2.23.

Definition II.2.25. We say that M' is a (μ, σ)*-limit* iff there is some $M \in \mathcal{K}$ such that M' is a (μ, σ)-limit over M.

Notation II.2.26. (1) For μ a cardinal and σ a limit ordinal with $\sigma < \mu^+$, we write $\mathcal{K}^{\sigma}_{\mu}$ for the collection of (μ, σ)-limit models of $\mathcal{K}$.

(2) We define

$$\mathcal{K}^*_\mu := \{M \in \mathcal{K} \mid M \text{ is a } (\mu, \theta)\text{−limit model for some limit ordinal } \theta < \mu^+\}.$$

as the *collection of limit models of* $\mathcal{K}$.

Limit models also exist in certain abstract elementary classes. By repeated applications of Lemma II.2.21, the existence of (μ, ω)-limit models can be proved:

Proposition II.2.27 (Theorem 1.3.1 from [ShVi]). *Let μ be a cardinal such that $\mu < \lambda$. For every $M \in \mathcal{K}^{am}_{\mu}$, there exists $M' \in \mathcal{K}$ such that $M \prec_{\mathcal{K}} M'$ and M' is a (μ, ω)-limit over M.*

In order to extend this argument further to yield the existence of (μ, σ)-limits for arbitrary limit ordinals $\sigma < \mu^+$, we need to be able to verify that limit models are in fact amalgamation bases. We will examine this in Section 2.4.

While the existence of certain limit models is relatively easy to derive from the categoricity assumption, the uniqueness of limit models is more difficult. Here we recall two easy uniqueness facts which state that limit models of the same length are isomorphic:

Proposition II.2.28 (Fact 1.3.6 from [ShVi]). *Let $\mu \geq LS(\mathcal{K})$ and $\sigma < \mu^+$. If M_1 and M_2 are (μ, σ)-limits over M, then there exists an isomorphism $g : M_1 \to M_2$ such that $g \restriction M = id_M$. Moreover if M_1 is a (μ, σ)-limit over M_0; N_1 is a (μ, σ)-limit over N_0 and $g : M_0 \cong N_0$, then there exists a $\prec_{\mathcal{K}}$-mapping, $\hat{g}$, extending g such that $\hat{g} : M_1 \cong N_1$.*

Proposition II.2.29 (Fact 1.3.7 from [ShVi]). *Let μ be a cardinal and σ a limit ordinal with $\sigma < \mu^+ \leq \lambda$. If M is a (μ, σ)-limit model, then M is a $(\mu, cf(\sigma))$-limit model.*

A more challenging uniqueness question is to prove that two limit models of different lengths ($\sigma_1 \neq \sigma_2$) are isomorphic (Conjecture II.1.2). A main result of this chapter, Theorem II.11.1, is a solution to this conjecture.

We will need one more notion of limit model, which will appear implicitly in the proofs of Theorem II.6.10, Theorem II.7.11, Theorem II.8.7 and Theorem II.9.7. This notion is a mild extension of the notion of limit models already defined:

Definition II.2.30. Let μ be a cardinal $< \lambda$, we say that $\check{M}$ is a (μ, μ^+) limit over M iff there exists a $\prec_{\mathcal{K}}$-increasing and continuous chain of models $\langle M_i \in \mathcal{K}^{am}_\mu \mid i < \mu^+ \rangle$ satisfying

(1) $M_0 = M$

(2) $\bigcup_{i<\mu^+} M_i = \check{M}$ and

(3) for $i < \mu^+$, M_{i+1} is universal over M_i

Remark II.2.31. While it is known that (μ, θ)-limit models are amalgamation bases when $\theta < \mu^+$, it is open as to whether or not (μ, μ^+)-limits are amalgamation bases. To avoid confusion between these two concepts of limit models, we will always denote (μ, μ^+)-limit models with a ˇ above the model's name (ie. $\check{M}$).

The existence of (μ, μ^+)-limit models follows from the fact that (μ, θ)-limit models are amalgamation bases when $\theta < \mu^+$, see Corollary II.4.9. The uniqueness of (μ, μ^+)-limit models (Proposition II.2.32) can be shown using an easy back and forth construction as in the proof of Proposition II.2.28.

Proposition II.2.32. *Suppose $\check{M}_1$ and $\check{M}_2$ are (μ, μ^+)-limits over M_1 and M_2, respectively. If there exists an isomorphism $h : M_1 \cong M_2$, then h can be extended to an isomorphism $g : \check{M}_1 \cong \check{M}_2$.*

(μ, μ^+)-limit models turn to be useful as replacement for monster models as Proposition II.2.32 and the following proposition provide some level of homogeneity:

Proposition II.2.33. *If $\check{M}$ is a (μ, μ^+)-limit, then for every $N \prec_{\mathcal{K}} \check{M}$ with $N \in \mathcal{K}^{am}_\mu$, we have that $\check{M}$ is universal over N. Moreover, $\check{M}$ is a (μ, μ^+)-limit over N.*

2.3 Ehrenfeucht-Mostowski Models

Since $\mathcal{K}$ has no maximal models, $\mathcal{K}$ has models of cardinality $Hanf(\mathcal{K})$. Then by Theorem II.3.1, we can construct Ehrenfeucht-Mostowski models.

Theorem II.3.1 (Claim 0.6 of [Sh 394] or see [Gr]). *Assume that $\mathcal{K}$ is an AEC that contains a model of cardinality $\geq \beth_{(2^{LS(\mathcal{K})})^+}$. Then, there is a Φ, proper for linear orders, such that for linear orders $I \subseteq J$ we have that*

(1) $EM(I,\Phi) \restriction L(\mathcal{K}) \prec_{\mathcal{K}} EM(J,\Phi) \restriction L(\mathcal{K})$ and

(2) $\|EM(I,\Phi) \restriction L(\mathcal{K})\| = |I| + LS(\mathcal{K})$.

We describe an index set which appears often in work toward the categoricity conjecture. This index set was used in [KoSh], [Sh 394] and [ShVi].

Notation II.3.2. Let $\alpha < \lambda$ be given. We define

$$I_\alpha := \left\{ \ \eta \in {}^\omega\alpha : \{n < \omega \mid \eta[n] \neq 0\} \text{ is finite } \right\}$$

Associate with I_α the lexicographical ordering $\lessdot$. If $X \subseteq \alpha$, we write $I_X := \big\{\eta \in {}^\omega\mathrm{X} : \{n < \omega \mid \eta[n] \neq 0\} \text{ is finite}\big\}$.

The following proposition is proved in several papers e.g. [ShVi].

Proposition II.3.3. *If $M \prec_{\mathcal{K}} EM(I_\lambda,\Phi) \restriction L(\mathcal{K})$ is a model of cardinality μ^+ with $\mu^+ < \lambda$, then there exists a $\prec_{\mathcal{K}}$-mapping $f : M \to EM(I_{\mu^+},\Phi) \restriction L(\mathcal{K})$.*

A variant of this universality property is (implicit in Lemma 3.7 of [KoSh]):

Proposition II.3.4. *Suppose κ is a regular cardinal. If $M \prec_{\mathcal{K}} EM(I_\kappa,\Phi) \restriction L(\mathcal{K})$ is a model of cardinality $< \kappa$ and $N \prec_{\mathcal{K}} EM(I_\lambda,\Phi) \restriction L(\mathcal{K})$ is an extension of M of cardinality $\|M\|$, then there exists a $\prec_{\mathcal{K}}$-embedding $f : N \to EM(I_\kappa,\Phi) \restriction L(\mathcal{K})$ such that $f \restriction M = id_M$.*

2.4 Amalgamation Bases

Since the amalgamation property for abstract elementary classes is inherent in the definition of types, most work towards understanding AECs has been under the assumption that the class $\mathcal{K}$ has the amalgamation property. In [ShVi], Shelah and Villaveces begin to tackle the categoricity problem with an approach that does not require the amalgamation property as an assumption. Shelah and Villaveces, however, prove a weak amalgamation property, which they refer to as *density of amalgamation bases*, summarized here:

Theorem II.4.1 (Theorem 1.2.4 from [ShVi]). *For every $M \in \mathcal{K}_{<\lambda}$, there exists $N \in \mathcal{K}^{am}_{\|M\|}$ with $M \prec_{\mathcal{K}} N$.*

We can now improve Lemma II.2.21 slightly. This improvement is used throughout this paper.

Lemma II.4.2. *For every μ with $LS(\mathcal{K}) < \mu < \lambda$, if $M \in \mathcal{K}^{am}_\mu$, $N \in \mathcal{K}$ and $\bar{a} \in {}^{\mu^+>}\mathrm{N}$ are such that $M \prec_{\mathcal{K}} N$, then there exists $M^{\bar{a}} \in \mathcal{K}^{am}_\mu$ such that $M^{\bar{a}}$ is universal over M and $M \bigcup \bar{a} \subseteq M^{\bar{a}}$.*

Proof. By Axiom 6 of AEC, we can find $M' \prec_{\mathcal{K}} N$ of cardinality μ containing $M \bigcup \bar{a}$. Applying Theorem II.4.1, there exists an amalgamation base of cardinality μ, say M'', extending M'. By Lemma II.2.21 we can find a universal extension of M'' of cardinality μ, say $M^{\bar{a}}$.

Notice that $M^{\bar{a}}$ is also universal over M. Why? Suppose M^* is an extension of M of cardinality μ. Since M is an amalgamation base we can amalgamate M'' and M^* over M. WLOG we may assume that the amalgam, M^{**}, is an extension of M'' of cardinality μ and $f^* : M^* \rightarrow M^{**}$ with $f^* \restriction M = id_M$.

$$\begin{array}{ccc} M^* & \xrightarrow[f^{**}]{} & M^{**} \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle id} \\ M & \xrightarrow[id]{} & M'' \end{array}$$

Now, since $M^{\bar{a}}$ is universal over M'', there exists a $\prec_{\mathcal{K}}$-mapping g such that $g : M^{**} \to M^{\bar{a}}$ with $g \restriction M'' = id_{M''}$. Notice that $g \circ f^*$ gives us the desired mapping of M^* into $M^{\bar{a}}$. $\dashv$

While Theorem II.4.1 asserts the existence of amalgamation bases, it is unknown (in this context) what characterizes amalgamation bases. Shelah and Villaveces have claimed that every limit model is an amalgamation base (Fact 1.3.10 of [ShVi]), using $\diamondsuit_{S^{\mu^+}_{\mathrm{cf}(\mu)}}$. Notice this is more than the asssumption of GCH that they make throughout their paper. We believe that $\diamondsuit_{S^{\mu^+}_{\mathrm{cf}(\mu)}}$ is not sufficient to carry out the argument that they suggest. A stronger set theoretic assumption (namely the weak form of diamond listed as Assumption II.1.1.(5)) is needed. We provide a proof that every (μ, θ)-limit model with $\theta < \mu^+$ is an amalgamation base under this additional assumption:

Theorem II.4.3. *Under Assumption II.1.1 (specifically under the set theoretic assumption of $\Phi_{\mu^+}(S^{\mu^+}_\theta)$ for every regular $\theta < \mu^+$), if M is a (μ, θ)-limit for some θ with $\theta < \mu^+ \leq \lambda$, then M is an amalgamation base.*

Let us first recall some set theoretic definitions and facts concerning the weak diamond.

Definition II.4.4. Let θ be a regular ordinal $< \mu^+$. We denote

$$S^{\mu^+}_\theta := \{\alpha < \mu^+ \mid \mathrm{cf}(\alpha) = \theta\}.$$

Definition II.4.5. For μ a cardinal and $S \subseteq \mu^+$ a stationary set, $\Phi_{\mu^+}(S)$ is said to hold iff for all $F : {}^{\lambda^+>}2 \rightarrow 2$ there exists $g : \lambda^+ \rightarrow 2$ so that for every $f : \lambda^+ \rightarrow 2$ the set

$$\{\delta \in S \mid F(f \upharpoonright \delta) = g(\delta)\} \text{ is stationary.}$$

We will be using a consequence of $\Phi_{\mu^+}(S)$, called $\Theta_{\mu^+}(S)$ (see [Gr]).

Definition II.4.6. For μ a cardinal $S \subseteq \mu^+$ a stationary set, $\Theta_{\mu^+}(S)$ is said to hold if and only if for all families of functions

$$\{f_\eta \ : \ \eta \in {}^{\mu^+}2 \text{ where } f_\eta : \mu^+ \rightarrow \mu^+\}$$

and for every club $C \subseteq \mu^+$, there exist $\eta \neq \nu \in {}^{\mu^+}2$ and there exists a $\delta \in C \cap S$ such that

(1) $\eta \upharpoonright \delta = \nu \upharpoonright \delta$,

(2) $f_\eta \upharpoonright \delta = f_\nu \upharpoonright \delta$ and

(3) $\eta[\delta] \neq \nu[\delta]$.

The following implications (Fact II.4.7) are a consequence of work of Devlin and Shelah [DS]. For an exposition of Fact II.4.7 see [Gr].

Fact II.4.7. $2^\mu < 2^{\mu^+} \implies \Phi_{\mu^+}(S^{\mu^+}_\theta) \implies \Theta_{\mu^+}(S^{\mu^+}_\theta)$.

Before we begin the proof of Theorem II.4.3, notice that:

Remark II.4.8 (Invariance). By Axiom 1 of AEC, if M is an amalgamation base and f is an $\prec_{\mathcal{K}}$-embedding, then $f(M)$ is an amalgamation base.

Proof of Theorem II.4.3. Given μ, suppose that θ is the minimal infinite ordinal $< \mu^+$ such that there exists a model M which is a (μ, θ)-limit and not an amalgamation base. Notice that by Proposition II.2.29, we may assume that $\mathrm{cf}(\theta) = \theta$.

Now we define by induction on the length of $\eta \in {}^{\mu^+>}2$ a tree of structures, $\langle M_\eta \mid \eta \in {}^{\mu^+>}2\rangle$, satisfying:

(1) for $\eta \lessdot \nu \in {}^{\mu^+>}2$, $M_\eta \prec_{\mathcal{K}} M_\nu$

(2) for $l(\eta)$ a limit ordinal with $\mathrm{cf}(l(\eta)) \leq \theta$, $M_\eta = \bigcup_{\alpha<l(\eta)} M_{\eta\restriction\alpha}$

(3) for $\eta \in {}^{\alpha}2$ with $\alpha \in S^{\mu^+}_\theta$,

  (a) M_η is a (μ, θ)-limit model

  (b) $M_{\eta\hat{}0}, M_{\eta\hat{}1}$ cannot be amalgamated over M_η

  (c) $M_{\eta\hat{}0}$ and $M_{\eta\hat{}1}$ are amalgamation bases of cardinality μ

(4) for $\eta \in {}^{\alpha}2$ with $\alpha \notin S^{\mu^+}_\theta$,

  (a) M_η is an amalgamation base

  (b) $M_{\eta\hat{}0}, M_{\eta\hat{}1}$ are universal over M_η and

  (c) $M_{\eta\hat{}0}$ and $M_{\eta\hat{}1}$ are amalgamation bases of cardinality μ (it may be that $M_{\eta\hat{}0} = M_{\eta\hat{}1}$ in this case).

This construction is possible:

$\eta = \langle\rangle$: By Theorem II.4.1, we can find $M' \in \mathcal{K}^{am}_\mu$ such that $M \prec_{\mathcal{K}} M'$. Define $M_{\langle\rangle} := M'$.

$l(\eta)$ is a limit ordinal: When $\mathrm{cf}(l(\eta)) > \theta$, let $M'_\eta := \bigcup_{\alpha<l(\eta)} M_{\eta\restriction\alpha}$. M'_η is not necessarily an amalgamation base, but for the purposes of this construction, continuity at such limits is not important. Thus we can find an extension of M'_η, say M_η, of cardinality μ where M_η is an amalgamation base.

For η with $\mathrm{cf}(l(\eta)) \leq \theta$, we require continuity. Define $M_\eta := \bigcup_{\alpha<l(\eta)} M_{\eta\restriction\alpha}$. We need to verify that if $l(\eta) \notin S^{\mu^+}_\theta$, then M_η is an amalgamation base. In fact,

we will show that such a M_η will be a $(\mu, \mathrm{cf}(l(\eta)))$-limit model. Let $\langle \alpha_i \mid i < \mathrm{cf}(l(\eta))\rangle$ be an increasing and continuous sequence of ordinals converging to $l(\eta)$ such that $\mathrm{cf}(\alpha_i) < \theta$ for every $i < \mathrm{cf}(l(\eta))$. Condition (4b) guarantees that for $i < \mathrm{cf}(l(\eta))$, $M_{\eta\restriction\alpha_{i+1}}$ is universal over $M_{\eta\restriction\alpha}$. Additionally, condition (2) ensures us that $\langle M_{\eta\restriction\alpha_i} \mid i < \mathrm{cf}(l(\eta))\rangle$ is continuous. This sequence of models witnesses that M_η is a $(\mu, \mathrm{cf}(l(\eta)))$-limit model. By our minimal choice of θ, we have that $(\mu, \mathrm{cf}(l(\eta)))$-limit models are amalgamation bases.

$\eta\hat{}i$ where $l(\eta) \in S^{\mu^+}_\theta$: We first notice that $M_\eta := \bigcup_{\alpha<l(\eta)} M_{\eta\restriction\alpha}$ is a (μ,θ)-limit model. Why? Since $l(\eta) \in S^{\mu^+}_\theta$ and θ is regular, we can find an increasing and continuous sequence of ordinals, $\langle \alpha_i \mid i < \theta\rangle$ converging to $l(\eta)$ such that for each $i < \theta$ we have that $\mathrm{cf}(\alpha_i) < \theta$. Condition (4b) of the construction guarantees that for each $i < \theta$, $M_{\eta\restriction\alpha_{i+1}}$ is universal over $M_{\eta\restriction\alpha_i}$. Thus $\langle M_{\eta\restriction\alpha_i} \mid i < \theta\rangle$ witnesses that M_η is a (μ,θ)-limit model.

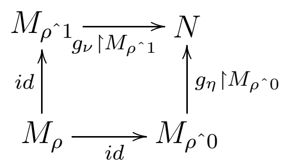
Since M_η is a (μ,θ)-limit, we can fix an isomorphism $f : M \cong M_\eta$. By Remark II.4.8, M_η is not an amalgamation base. Thus there exist $M_{\eta\hat{}0}$ and $M_{\eta\hat{}1}$ extensions of M_η which cannot be amalgamated over M_η. WLOG we can choose, $M_{\eta\hat{}0}$ and $M_{\eta\hat{}1}$ to be elements of $\mathcal{K}^{am}_\mu$.

$\eta\hat{}i$ where $l(\eta) \notin S^{\mu^+}_\theta$: Since M_η is an amalgamation base, we can choose $M_{\eta\hat{}0}$ and $M_{\eta\hat{}1}$ to be extensions of M_η such that $M_{\eta\hat{}l} \in \mathcal{K}^{am}_\mu$ and $M_{\eta\hat{}l}$ is universal over M_η, for $l = 0, 1$.

This completes the construction. For every $\eta \in {}^{\mu^+}2$, define $M_\eta := \bigcup_{\alpha<\mu^+} M_{\eta\restriction\alpha}$. By categoricity in λ and Proposition II.3.3, we can fix a $\prec_{\mathcal{K}}$-mapping $g_\eta : M_\eta \to EM(I_{\mu^+}, \Phi) \restriction L(\mathcal{K})$ for each $\eta \in {}^{\mu^+}2$. Now apply $\Theta_{\mu^+}(S^{\mu^+}_\theta)$ to find $\eta, \nu \in {}^{\mu^+}2$ and $\alpha \in S^{\mu^+}_\theta$ such that

- $\rho := \eta \restriction \alpha = \nu \restriction \alpha$,
- $\eta[\alpha] = 0$, $\nu[\alpha] = 1$ and
- $g_\eta \restriction M_\rho = g_\nu \restriction M_\rho$.

By Axiom 6 (the Löwenheim-Skolem property) of AEC, there exists $N \prec_{\mathcal{K}} EM(I_{\mu^+}, \Phi) \restriction L(\mathcal{K})$ of cardinality μ such that the following diagram commutes:



Notice that $g_\eta \restriction M_{\rho\hat{}0}$, $g_\nu \restriction M_{\rho\hat{}1}$ and N witness that $M_{\rho\hat{}0}$ and $M_{\rho\hat{}1}$ can be amalgamated over M_ρ. Since $l(\rho) = \alpha \in S^{\mu^+}_\theta$, we contradict condition (3b) of the construction.

$\dashv$

Corollary II.4.9 (Existence of limit models and (μ, μ^+)-limit models). *For every cardinal μ and limit ordinal θ with $\theta \leq \mu^+ \leq \lambda$, if M is an amalgamation base of cardinality μ, then there exists $M' \in \mathcal{K}^{am}_\mu$ which is a (μ, θ)-limit over M.*

Proof. By repeated applications of Lemma II.2.21 and Theorem II.4.3. $\dashv$

2.5 Weak Disjoint Amalgamation

Shelah and Villaveces prove a version of weak disjoint amalgamation in an attempt to prove an extension property for towers. We will be using weak disjoint amalgamation to build extensions of towers. We provide a proof of weak disjoint amalgamation here for completeness.

Theorem II.5.1 (Weak Disjoint Amalgamation [ShVi]). *Given $\lambda > \mu \geq LS(\mathcal{K})$ and $\alpha, \theta_0 < \mu^+$ with θ_0 regular. If M_0 is a (μ, θ_0)-limit and $M_1, M_2 \in \mathcal{K}_\mu$ are $\prec_{\mathcal{K}}$-extensions of M_0, then for every $\bar{b} \in {}^{\alpha}(M_1 \backslash M_0)$, there exist M_3, a model, and h, a $\prec_{\mathcal{K}}$-embedding, such that*

(1) $h : M_2 \to M_3$;

(2) $h \restriction M_0 = id_{M_0}$ and

(3) $h(M_2) \cap \bar{b} = \emptyset$ (equivalently $h(M_2) \cap M_1 = M_0$).

Shelah and Villaveces provide a proof of this theorem in [ShVi]. It has been suggested that I elaborate on the proof here. John Baldwin may have a simplification of this proof.

Proof. Suppose that M_0, M_1, M_2 and $\bar{b} \in M_1$ form a counter-example. Since M_0 is a μ amalgamation base, we may assume that there exists $M^* \in \mathcal{K}_\mu$ with $M_1, M_2 \prec_{\mathcal{K}} M^*$. Let θ be regular and $< \mu^+$ such that M_0 is a (μ, θ)-limit. We define a $\prec_{\mathcal{K}}$-increasing and continuous sequence of models $\langle N_i \mid i < \mu^+ \rangle$ satisfying:

(1) $N_i \in \mathcal{K}^{am}_\mu$

(2) N_{i+1} is universal over N_i and

(3) when $\mathrm{cf}(i) = \theta$, we additionally define N^1_i, N^2_i, N^*_i and $\bar{b}_i \in N^1_i$ such that there exists an isomorphism $f_i : M^* \cong N^*_i$ with $f_i(M_0) = N_i$, $f_i(M_1) = N^1_i$, $f_i(M_2) = N^2_i$ and $f_i(\bar{b}) = \bar{b}_i$.

The construction is possible by Lemma II.2.21, Theorem II.4.3 and Proposition II.2.28.

Let $N_{\mu^+} := \bigcup_{i<\mu^+} N_i$. Since $\mathcal{K}$ is categorical in λ, Proposition II.3.3 allows us to find a $\prec_{\mathcal{K}}$-mapping $g : N_{\mu^+} \to EM(I^+_\mu, \Phi) \restriction L(\mathcal{K})$. So WLOG, we may assume that $N_{\mu^+} \prec_{\mathcal{K}} EM(I^+_\mu, \Phi) \restriction L(\mathcal{K})$.

Let $E \subseteq \mu^+$ be a club such that

$$\delta \in E \Rightarrow N_\delta \prec_{\mathcal{K}} EM(I_\delta, \Phi) \restriction L(\mathcal{K}).$$

For each $i \in S^{\mu^+}_\theta$, choose a Skolem-term τ_i and a sequence of indices $\alpha_{i,0}, \dots, \alpha_{i,n_i-1}$ such that $\bar{b}_i = \tau_i(\alpha_{i,0}, \dots, \alpha_{i,n_i-1})$. Let $m_i < n_i$ be such

$$k < m_i \Leftrightarrow \alpha_{i,k} \in I_i.$$

Set $\alpha_{i,<m_i} := \langle \alpha_{i,k} \mid 0 \leq k < m_i \rangle$ and $\alpha_{i,\geq m_i} := \langle \alpha_{i,k} \mid m_i \leq k < n_i \rangle$.

Let $\delta_0 \in E \cap S^{\mu^+}_\theta$.

For every δ_1, with $\delta_0 < \delta_1 < \mu^+$. Define g_{δ_1} to be the $\prec_{\mathcal{K}}$-mapping from $EM(I_{\delta_1}, \Phi) \restriction L(\mathcal{K})$ to $EM(I_{\mu^+}, \Phi) \restriction L(\mathcal{K})$ induced by the mapping from μ^+ to μ^+ defined by

$$j \mapsto \begin{cases} j & \text{if } j < \delta_0 \\ \delta_1 + j & \text{if } \delta_0 \leq j < \delta_1 \end{cases}$$

Let $\delta \in C$ with $\delta_0 < \delta$.

Subclaim II.5.2. *Then* $g_{\delta_1}(N^1_{\delta_0}) \cap \bar{b}_{\delta_0} = \emptyset$.

Proof. Suppose the claim fails. Then there exist $b \in \bar{b}_{\delta_0}$, a Skolem term σ_δ and a sequence of elements of I_δ

$$\beta_{\delta,0}, \dots, \beta_{\delta,m_\delta-1}, \beta_{\delta,m_\delta}, \dots, \beta_{\delta,n_\delta-1}$$

such that

$$k < m_\delta \Leftrightarrow \beta_{\delta,k} \in I_{\delta_0}$$

and $b = \sigma_\delta(\beta_{\delta,0}, \ldots, \beta_{\delta,n_\delta-1})$.

Let $\beta_{\delta,<m_\delta} := \langle \beta_{\delta,k} \mid 0 \leq k < m_\delta \rangle$ and $\beta_{\delta,\geq m_\delta} := \langle \beta_{\delta,k} \mid m_\delta \leq k < n_\delta \rangle$.

Notice that

$$EM(I_{\mu^+}, \Phi) \restriction L(\mathcal{K}) \models b = \sigma_{\delta_0}(\beta_{\delta,<m_\delta}; \beta_{\delta,\geq m_\delta}) = \tau_{\delta_0}(\alpha_{\delta_0,<m_{\delta_0}}; \alpha_{\delta_0,\geq m_{\delta_0}}).$$

Since all our indices are finite sequences and δ_0 is a limit ordinal, there exists $\delta^* < \delta_0$ and such that $\alpha_{\delta_0,<m_{\delta_0}}, \beta_{\delta,<m_\delta} \in I_{\delta^*}$. This allows us to find a sequence $\alpha^*\hat{}\beta^* \in I_{\delta_0}$ which has the same type over I_{δ^*} (with respect to the lexicographical ordering) as $\alpha_{\delta_0,\geq m_{\delta_0}}\hat{}\beta_{\delta,\geq m_\delta}$. So by indiscernibility

$$(*) \quad EM(I_{\mu^+}, \Phi) \restriction L(\mathcal{K}) \models \sigma_{\delta_0}(\beta_{\delta,<m_\delta}; \beta^*) = \tau_{\delta_0}(\alpha_{\delta_0,<m_{\delta_0}}; \alpha^*).$$

By our definition of g_δ, we have that

$$(*)_\delta \quad k \geq m_\delta \Leftrightarrow \beta_{\delta,k} \in I_{\delta \setminus \delta_1 \cup \delta_0}.$$

In other words when $k \geq m_\delta$, every term from the sequence $\beta_{\delta,k}$ which is larger than δ_0 is also larger than δ_1. Thus, for $k \geq m_\delta$, the ordinals in $\beta_{\delta,k}$ above δ_0 are all greater than the ordinals above δ_0 appearing in the sequences $\alpha_{\delta_0 \geq m_{\delta_0}}$, α^* and $\beta_{\delta,<m_\delta}$. Thus the type (with respect to the lexicographical ordering) of $\beta_{\delta,\geq m_\delta}$ and β^* are the same over $\alpha_{\delta,<m_{\delta_0}}\hat{}\alpha^*\hat{}\beta_{\delta,<m_\delta}$. Indiscernibility applied to $(*)$ yields:

$$EM(I_{\mu^+}, \Phi) \restriction L(\mathcal{K}) \models \sigma_{\delta_0}(\beta_{\delta,<m_\delta}; \beta_{\delta,\geq m_\delta}) = \tau_{\delta_0}(\alpha_{\delta_0,<m_{\delta_0}}; \alpha^*).$$

Notice that $\sigma_{\delta_0}(\beta_{\delta,<m_\delta};\beta_{\delta,\geq m_\delta}) = b$. Thus we have found a way to construct b from I_{δ_0} (by $\tau_{\delta_0}(\alpha_{\delta_0,<m_{\delta_0}};\alpha^*)$). This contradicts our choice of $b \notin EM(I_{\delta_0}) \restriction L(\mathcal{K})$.

$\dashv$

Let δ_1 be as in Subclaim II.5.2. There exists an ordinal $\alpha_2 < \mu^+$ such that $g_{\delta_1} : \delta_1 \to \alpha_2$. Let g be the $\prec_{\mathcal{K}}$-mapping induced by g_{δ_1} such that $g : N_{\delta_1} \to EM(I_{\alpha_2}, \Phi) \restriction L(\mathcal{K})$. Notice that by our choice of δ_1, we have that g and $EM(I_{\alpha_2}, \Phi) \restriction L(\mathcal{K})$ witnesses that $N_{\delta_0}, N^1_{\delta_0}, N^2_{\delta_0}$ and $\bar{b}_{\delta_0}$ can be weakly disjointly amalgamated.

$\dashv$

Let us state an easy corollary of Theorem II.5.1 that will simplify future constructions:

Corollary II.5.3. *Suppose μ, M_0, M_1, M_2 and $\bar{b}$ are as in the statement of Theorem II.5.1. If $\check{M}$ is universal over M_1, then there exists a $\prec_{\mathcal{K}}$-mapping h such that*

(1) $h : M_2 \to \check{M}$,

(2) $h \restriction M_0 = id_{M_0}$ and

(3) $h(M_2) \cap \bar{b} = M_0$ (equivalently $h(M_2) \cap M_1 = \emptyset$).

Proof. By Theorem II.5.1, there exists a $\prec_{\mathcal{K}}$-mapping g and a model M_3 of cardinality μ such that

- $g : M_2 \to M_3$
- $g \restriction M_0 = id_{M_0}$

- $g(M_2) \cap \bar{b} = M_0$ and
- $M_1 \prec_{\mathcal{K}} M_3$.

Since $\check{M}$ is universal over M_1, we can fix a $\prec_{\mathcal{K}}$-mapping f such that

- $f : M_3 \to \check{M}$ and
- $f \restriction M_1 = id_{M_1}$

Notice that $h := g \circ f$ is the desired mapping from M_2 into $\check{M}$.

$\dashv$

2.6 $<^b_{\mu,\alpha}$-extension property for $\mathcal{K}^*_{\mu,\alpha}$

Shelah introduced towers in [Sh 48] and [Sh 87b] as a tool to build a model of cardinality μ^+ from models of cardinality μ. Here we will use the towers to prove the uniqueness of limit models by producing a model which is simultaneously a (μ, θ_1)-limit model and a (μ, θ_2)-limit model. The construction of such a model is sufficient to prove the uniqueness of limit models by Proposition II.2.28.

Definition II.6.1 (Towers Definition 3.1.1 of [ShVi]). Let $\mu > LS(\mathcal{K})$ and $\alpha, \theta < \mu^+$

(1)

$$\mathcal{K}_{\mu,\alpha} := \left\{ (\bar{M}, \bar{a}) \;\middle|\; \begin{array}{l} (\bar{M}, \bar{a}) := (\langle M_\gamma \mid \gamma < \alpha \rangle, \langle a_\gamma \mid \gamma < \alpha \rangle); \\ \bar{M} \text{ is } \prec_{\mathcal{K}} -\text{increasing}; \\ \text{for every } \gamma < \alpha,\ a_\gamma \in M_{\gamma+1} \backslash M_\gamma; \\ \text{for every } \gamma < \alpha,\ M_\gamma \in \mathcal{K}_\mu \end{array} \right\}$$

(2) $\mathcal{K}^\theta_{\mu,\alpha} := \{(\bar{M}, \bar{a}) \in \mathcal{K}_{\mu,\alpha} \mid \text{for every } \gamma < \alpha,\ M_\gamma \text{ is a } (\mu, \theta)\text{-limit}\}$

(3) $\mathcal{K}^*_{\mu,\alpha} := \bigcup_{\theta<\mu^+} \mathcal{K}^\theta_{\mu,\alpha}$

Fact II.6.2 (Fact 3.1.7 from [ShVi]). *Suppose $\mathcal{K}$ is categorical in λ. Given $\lambda > \mu \geq LS(\mathcal{K})$, $\alpha < \mu^+$ and θ a regular cardinal with $\theta < \mu^+$, we have that $\mathcal{K}^\theta_{\mu,\alpha} \neq \emptyset$.*

Roughly speaking, we will construct an array of models of width σ_1 and height σ_2 in such a way that the union will simultaneously be a (μ, σ_1)-limit model and a (μ, σ_2)-limit model. Each row in our array will be a tower from $\mathcal{K}^*_{\mu,\theta_1}$. We define the array by induction on the heigth (σ_2) by finding an "increasing" and continuous chain of towers from $\mathcal{K}^*_{\mu,\theta_1}$. We need to make explicit what we mean by "increasing." One property that the ordering on towers should have is that the union of an "increasing" chain of towers from $\mathcal{K}^*_{\mu,\theta_1}$ should also be a member of $\mathcal{K}^*_{\mu,\theta_1}$. In particular we need to guarantee that the models that appear in the union be limit models. This motivates the following ordering on towers:

Definition II.6.3 (Definition 3.1.3 of [ShVi]). For $(\bar{M}, \bar{a}), (\bar{N}, \bar{b}) \in \mathcal{K}^*_{\mu,\alpha}$ we say that

(1) $(\bar{M}, \bar{a}) \leq^b_{\mu,\alpha} (\bar{N}, \bar{b})$ if and only if

   (a) $\bar{a} = \bar{b}$;

   (b) for every $\gamma < \alpha$, $M_\gamma \preceq_{\mathcal{K}} N_\gamma$ and

   (c) whenever $M_\gamma \prec_{\mathcal{K}} N_\gamma$, then N_γ is universal over M_γ.

(2) $(\bar{M}, \bar{a}) <^b_{\mu,\alpha} (\bar{N}, \bar{b})$ if and only if $(\bar{M}, \bar{a}) \leq^b_{\mu,\alpha} (\bar{N}, \bar{b})$ and for every $\gamma < \alpha$, $M_\gamma \neq N_\gamma$.

Notation II.6.4. We will often be looking at extension of an initial segment of a tower. We introduce the following notation for this. Suppose $(\bar{M}, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$.

Let $\beta < \alpha$. We write $(\bar{M}, \bar{a}) \restriction \beta$ for the tower $(\langle M_i \mid i < \beta\rangle, \langle a_i \mid i < \beta\rangle) \in \mathcal{K}^*_{\mu,\beta}$. We also abbreviate $\langle a_i \mid i < \beta\rangle$ by $\bar{a} \restriction \beta$.

Remark II.6.5. If $\langle(\bar{M}, \bar{a})_\sigma \in \mathcal{K}^*_{\mu,\alpha} \mid \sigma < \gamma\rangle$ is a $<^b_{\mu,\alpha}$-increasing and continuous chain with $\gamma < \mu^+$, then $\bigcup_{\sigma<\gamma}(\bar{M}, \bar{a})_\sigma \in \mathcal{K}^*_{\mu,\alpha}$. Why? Notice that for $i < \alpha$, $M_{i,\gamma} := \bigcup_{\sigma<\gamma} M_{i,\sigma}$ is a limit model, witnessed by $\langle M_{i,\sigma} \mid \sigma < \gamma\rangle$.

In order to construct a non-trivial chain of towers, we need to be able to take proper $<^b_{\mu,\alpha}$-extensions.

Definition II.6.6. We say the $<^b_{\mu,\alpha}$-*extension property* holds iff for every $(\bar{M}, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ there exists $(\bar{M}', \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ such that $(\bar{M}, \bar{a}) <^b_{\mu,\alpha} (\bar{M}', \bar{a})$.

Remark II.6.7. Shelah and Villaveces claim the $<^b_{\mu,\alpha}$-extension property as Fact 3.19(1) in [ShVi]. Their proof does not converge. As of the Fall of 2001, they were unable to produce a proof of this claim.

We will prove the $<^b_{\mu,\alpha}$-extension property for a particular class of towers:

Definition II.6.8. $(\langle M_i \mid i < \alpha\rangle, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ is *nice* provided that for every limit ordinal $i < \alpha$, we have that $\bigcup_{j<i} M_j$ is an amalgamation base.

Remark II.6.9. If $(\bar{M}, \bar{a})$ is continuous, then $(\bar{M}, \bar{a})$ is nice.

Notice that in the definition of towers, we do not require continuity at limit ordinals i of the sequence of models. This allows for towers in which $M_i \neq \bigcup_{j<i} M_j$. Since we only require that M_i is an amalgamation base, there are towers which are not necessarily nice. Moreover, the union of a $<^b$-increasing chain of $< \mu^+$ nice towers, is not necessarily nice.

Theorem II.6.10 (The $<^b_{\mu,\alpha}$-extension property for nice towers). *For every nice $(\bar{M}, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$, there exists a nice tower $(\bar{M}', \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ such that*

$(\bar{M},\bar{a}) <^b_{\mu,\alpha} (\bar{M}',\bar{a})$. *Moreover, if* $\bigcup_{i<\alpha} M_i$ *is an amalgamation base and* $\bigcup_{i<\alpha} M_i \prec_{\mathcal{K}} \check{M}$*, for some* (μ,μ^+)*-limit,* $\check{M}$*, then we can find a nice extension* $(\bar{M}',\bar{a})$ *such that* $\bigcup_{i<\alpha} M_i' \prec_{\mathcal{K}} \check{M}$*.*

It is natural to attempt to define $\langle M_i' \mid i < \alpha\rangle$ to form an extension $(\bar{M}',\bar{a})$ of $(\bar{M},\bar{a})$ by induction on $i < \alpha$ (as Shelah and Villaveces suggest). Theorem II.5.1 makes the base case possible. The limits could be taken care of by taking limits. The problem arises in the successor step. We would have defined M_i' extending M_i such that $M_i' \cap \{a_j \mid i \leq j < \alpha\} = \emptyset$. Theorem II.5.1 is too weak to find an extension *of both* M_i' *and* M_{i+1} which avoids $\{a_j \mid i+1 \leq j < \alpha\}$. We can only find M_{i+1}' which contains *an image of* M_i' and M_{i+1} and avoids $\{a_j \mid i+1 \leq j < \alpha\}$ by applying Theorem II.5.1 to M_{i+1}, some extension of $M_{i+1} \bigcup M_i'$, M_α and $\{a_j \mid i+1 \leq j < \alpha\}$.

Alternatively, one might try defining approximations $(\bar{M}',\bar{a}')^i \in \mathcal{K}^*_{\mu,i}$ a $<^b_{\mu,i}$-extension of $(\bar{M},\bar{a})$ by induction. In this construction, we have no problem with the successor stages (because we do not require the approximations to be increasing). However, we will get stuck at the limit stages, because we can no longer take unions.

Since Thoerem II.5.1 gives us a mapping from M_i' to M_{i+1}' we have decided to look at a directed system of models $(\langle M_i' \mid i < \alpha\rangle, \langle f_{i,j}' \mid i \leq j < \alpha\rangle)$.

Before beginning the proof of Theorem II.6.10, we prove the following lemma which will be used in the successor stage of the construction.

Lemma II.6.11. *Suppose* $(\bar{M},\bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ *lies inside a* (μ,μ^+)*-limit model,* $\check{M}$*, that is* $\bigcup_{i<\alpha} M_i \prec_{\mathcal{K}} \check{M}$*. If* $(\bar{M}',\bar{a}') \in \mathcal{K}^*_{\mu,j+1}$ *for some* $j+1 < \alpha$ *is a partial extension of* $(\bar{M},\bar{a})$ *(ie* $(\bar{M},\bar{a}) \restriction \beta <^b_{\mu,j+2} (\bar{M}',\bar{a}')$*), then there exists a* $\mathcal{K}$*-mapping*

$f : M_\beta \to \check{M}$ such that $f \restriction M_j = id_{M_j}$ and there exists $M'_{j+1} \in \mathcal{K}^*_\mu$ so that $(\langle f(M'_i) \mid i \leq j\rangle^\frown\langle M'_{j+1}\rangle, \bar{a} \restriction (j+2))$ is a partial $<^b_{\mu,j+2}$ extension of $(\bar{M}, \bar{a})$.

Proof. Since M'_j and M_{j+1} are both $\prec_\mathcal{K}$-substructures of $\check{M}$, we can get M''_{j+1} (a first approximation to the desired M'_{j+1}) such that $M''_{j+1} \in \mathcal{K}^*_\mu$ is universal over M'_j and universal over M_{j+1}. How? By the Downward Löwenheim Skolem Axiom (Axiom 6) of AEC and the density of amalgamation bases (Theorem II.4.1), we can find an amalgamation base L of cardinality μ such that $M'_j, M_{j+1} \prec_\mathcal{K} L$. By Lemma II.2.21 and Corollary II.4.9, there exists M''_{j+1}, a (μ, ω)-limit over L.

Subclaim II.6.12. *M''_{j+1} is universal over M'_j and is universal over M_{j+1}.*

Proof. It suffices to show that when $L_0 \prec_\mathcal{K} L_1 \prec_\mathcal{K} L$ are amalgamation bases of cardinality μ, if L is universal over L_1, then L is universal over L_0. Let L' be an extension of L_0 of cardinality μ. Since L_0 is an amalgamation base, we can find an amalgam L'' such that the following diagram commmutes:

$$\begin{array}{ccc} L' & \xrightarrow[h]{} & L'' \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle id} \\ L_0 & \xrightarrow[id]{} & L_1 \end{array}$$

Since L is universal over L_1, there exists $g : L'' \to L$ with $g \restriction L_1 = id_{L_1}$. Notice that $g \circ h : L' \to L$ with $g \circ h \restriction L_0 = id_{L_0}$. $\dashv$

M''_{j+1} may serve us well if it does not contain any a_l for $j+1 \leq l < \alpha$, but this is not guaranteed. So we need to make an adjustment. Notice that $\check{M}$ is universal over M_{j+1}. Thus we can apply Corollary II.5.3 to M_{j+1}, M_α, M''_{j+1} and $\langle a_l \mid j+1 \leq l < \alpha\rangle$. This yields a $\prec_\mathcal{K}$-mapping f such that

- $f : M''_{j+1} \to \check{M}$

- $f \restriction M_{j+1} = id_{M_{j+1}}$ and
- $f(M''_{j+1}) \cap \{a_l \mid j+1 \leq l < \alpha\} = \emptyset$.

Set $M'_{j+1} := f(M''_{j+1})$. ⊣

Proof of Theorem II.6.10. Let μ be a cardinal and α a limit ordinal such that $\alpha < \mu^+ \leq \lambda$. Let a nice tower $(\bar{M}, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ be given. Denote by M_α a model in $\mathcal{K}^{am}_\mu$ extending $\bigcup_{i<\alpha} M_i$. As discussed above, we have decided to look at a directed system of models $(\langle M'_i \mid i < \alpha\rangle, \langle f'_{i,j} \mid i \leq j < \alpha\rangle)$, as opposed to an increasing sequence, such that at each stage $i \leq \alpha$:

(1) $(\langle f'_{j,i}(M'_j) \mid j \leq i\rangle, \bar{a} \restriction i)$ is a $<^b_{\mu,i}$-extension of $(\bar{M}, \bar{a}) \restriction i$

(2) M'_i is universal over M_i,

(3) M'_{i+1} is universal over $f'_{i,i+1}(M'_i)$ and

(4) $f'_{j,i} \restriction M_j = id_{M_j}$,

It may be useful at this point to refer to Section 2.2 concerning directed systems and direct limits. In order to carry out the construction at limit stages, we need to work inside of a fixed structure. Fix $\check{M}$ to be a (μ, μ^+)-limit model over M_α. We will simultaneously define a directed system $(\langle \check{M}_i \mid i \leq \alpha\rangle, \langle \check{f}_{i,j} \mid i \leq j < \alpha\rangle)$ extending $(\langle M'_i \mid i < \alpha\rangle, \langle f'_{i,j} \mid i < j < \alpha\rangle)$ such that:

(5) $M'_i \prec_{\mathcal{K}} \check{M}$,

(6) $f'_{j,i}$ can be extended to an automorphism of $\check{M}$, $\check{f}_{j,i}$, for $j \leq i$ and

(7) $(\langle \check{M}_j = \check{M} \mid j < i\rangle, \langle \check{f}_{k,j} \mid k \leq j < i\rangle)$ forms a directed system.

Notice that the M'_i's will not necessarily form an extension of the tower $(\bar{M}, \bar{a})$. Rather, for each $i < \alpha$, we find some image of $\langle M_j \mid j < i\rangle$ which will extend the initial segment of length i of $(\bar{M}, \bar{a})$ (see condition (1) of the construction).

The construction is possible:

$i = 0$: Since M_0 is an amalgamation base, we can find $M_0'' \in \mathcal{K}_\mu^*$ (a first approximation of the desired M_0') such that M_0'' is universal over M_0. By Corollary II.5.3 (applied to M_0, M_α, M_0'' and $\bar{a}$), we can find a $\prec_\mathcal{K}$-mapping $h : M_0'' \to \check{M}$ such that $h \restriction M_0 = id_{M_0}$ and $h(M_0'') \cap \bar{a} = \emptyset$. Set $M_0' := h(M_0'')$, $f_{0,0}' := id_{M_0'}$ and $\check{f}_{0,0} := id_{\check{M}}$.

$i = j + 1$: Let h and M_{j+1}'' be as in Lemma II.6.11. Set $M_{j+1}' := h(M_{j+1}'')$, $f_{j+1,j+1}' = id_{M_{j+1}'}$, $\check{f}_{j+1,j+1} = id_{\check{M}}$ and $f_{j,j+1}' := h \restriction M_j'$. Since $\check{M}$ is a (μ, μ^+)-limit over both M_j' and $f_{j,j+1}'(M_j')$, by Proposition II.2.32 we can extend $f_{j,j+1}'$ to an automorphism of $\check{M}$, denoted by $\check{f}_{j,j+1}$.

To guarantee that we have a directed system, for $k < j$, define $f_{k,j+1}' := f_{j,j+1}' \circ f_{k,j}'$ and $\check{f}_{k,j+1} := \check{f}_{j,j+1} \circ \check{f}_{k,j}$.

i is a limit ordinal: Suppose that $(\langle M_j' \mid j < i\rangle, \langle f_{k,j}' \mid k \leq j < i\rangle)$ and $(\langle \check{M} \mid j < i\rangle, \langle \check{f}_{k,j} \mid k \leq j < i\rangle)$ have been defined. Since they are both directed systems, we can take direct limits, but we want to chose the representations of the direct limits carefully:

Claim II.6.13. *We can choose direct limits $(M_i^*, \langle f_{j,i}^* \mid j \leq i\rangle)$ and $(\check{M}_i^*, \langle \check{f}_{j,i}^* \mid j \leq i\rangle)$ of $(\langle M_j' \mid j < i\rangle, \langle f_{k,j}' \mid k \leq j < i\rangle)$ and $(\langle \check{M} \mid j < i\rangle, \langle \check{f}_{k,j} \mid k \leq j < i\rangle)$ respectively such that*

(a) $M_i^* \prec_\mathcal{K} \check{M}_i^*$

(b) $\check{f}_{j,i}^*$ is an automorphism of $\check{M}_i^*$ for every $j \leq i$

(c) $\check{M}_i^* = \check{M}$

(d) $f_{j,i}^* \restriction M_j = id_{M_j}$ for every $j < i$.

Proof. We will first find direct limits which satisfy the first 3 conditions ((a)-(c)). Then we will make adjustments to them in order to find direct limits which satisfy conditions (a)-(d) in the claim.

By Lemma II.2.6 we may choose direct limits $(M_i^{**}, \langle f_{j,i}^{**} \mid j \leq i\rangle)$ and $(\check{M}_i^{**}, \langle \check{f}_{j,i}^{**} \mid j \leq i\rangle)$ such that $M_i^{**} \prec_{\mathcal{K}} \check{M}_i^{**}$. By Claim II.2.8 we have that for every $j \leq i$, $\check{f}_{j,i}^{**}$ is an automorphism and $\check{M}_i^{**} = \check{M}$. Notice that $(M_i^{**}, \langle f_{j,i}^{**} \mid j \leq i\rangle)$ and $(\check{M}_i^{**}, \langle \check{f}_{j,i}^{**} \mid j \leq i\rangle)$ form a direct limits satisfying the first three properties. However, condition (d) may not hold. However we do know that:

Subclaim II.6.14. *$\langle f_{j,i}^{**} \restriction M_j \mid j < i\rangle$ is increasing.*

Proof. Let $j < k < i$ be given. By construction

$$f'_{j,k} \restriction M_j = id_{M_j}.$$

An application of $f_{k,i}^{**}$ yields

$$f_{k,i}^{**} \circ f'_{j,k} \restriction M_j = f_{k,i}^{**} \restriction M_j.$$

By the definition of directed limits, we have

$$f_{j,i}^{**} \restriction M_j = f_{k,i}^{**} \circ f'_{j,k} \restriction M_j = f_{k,i}^{**} \restriction M_j.$$

This completes the proof of Subclaim II.6.14

⊣

We still have not finished the proof of Claim II.6.13. By the subclaim, we have that $g := \bigcup_{j<i} f_{j,i}^{**} \restriction M_j$ is a partial autmorphism of $\check{M}$ from $\bigcup_{j<i} M_j$ onto $\bigcup_{j<i} f_{j,i}^{**}(M_j)$. Since $\check{M}$ is a (μ, μ^+)-limit model and since $\bigcup_{j<i} M_j$ is an amalgamation base we can extend g to $G \in Aut(\check{M})$ by Proposition II.2.32.

Notice this is the point of the proof where we use the assumption of niceness when we observe that $\bigcup_{j<i} M_j$ is an amalgamation base.

Now consider the direct limit defined by $M^*_i := G^{-1}(M^{**}_i)$ with $\langle f^*_{j,i} := G^{-1} \circ f^{**}_{j,i} \mid j < i\rangle$ and $f^*_{i,i} = id_{M^*_i}$ and the direct limit $\check{M}^*_i := \check{M}$ with $\langle \check{f}^*_{j,i} := G^{-1} \circ \check{f}^{**}_{j,i} \mid j < i\rangle$ and $\check{f}^*_{i,i} := id_{N^*_i}$. Notice that $f^*_{j,i} \restriction M_j = G^{-1} \circ f^{**}_{j,i} \restriction M_j = id_{M_j}$ for $j < i$. This completes the proof of Claim II.6.13

$\dashv$

Our choice of $(M^*_i, \langle f^*_{j,i} \mid j \leq i\rangle)$ and $(\check{M}^*_i, \langle \check{f}^*_{j,i} \mid j \leq i\rangle)$ from Claim II.6.13 may not be enough to complete the limit step since M^*_i may contain a_j for some $i \leq j < \alpha$. So we need to apply weak disjoint amalgamation and find isomorphic copies of theses systems. By Condition (4) of the construction, notice that M^*_i is a (μ, i)-limit model witnessed by $\langle f^*_{j,i}(M'_j) \mid j < i\rangle$. Hence M^*_i is an amalgamation base. Since M^*_i and M_i both live inside of $\check{M}$, we can find $M''_i \in \mathcal{K}^*_\mu$ which is universal over M_i and universal over M^*_i. By Corollary II.5.3 applied to M_i, M_α, M''_i and $\langle a_l \mid l \leq i < \alpha\rangle$ we can find $h : M''_i \to \check{M}$ such that $h \restriction M_i = id_{M_i}$ and $h(M''_i) \cap \{a_l \mid i \leq l < \alpha\} = \emptyset$.

Set $M'_i := h(M''_i)$, $f'_{i,i} := id_{M_{i,i}}$, $\check{f}_{i,i} := id_{\check{M}}$ and for $j < i$, $f'_{j,i} := h \circ f^*_{j,i}$. We need to verify that for $j \leq i$, $f'_{j,i}(M'_j) \bigcap \{a_l \mid j \leq l < \alpha\} = \emptyset$. Clearly by our application of weak disjoint amalgamation, we have that for every l with $i \leq l < \alpha$ and every $j \leq i$, $a_l \notin f'_{j,i}(M'_j)$ since $M'_i \supseteq f'_{j,i}(M'_j)$. Suppose that $j < i$ and l is such that $j \leq l < i$. By construction $a_l \notin f'_{j,l+1}(M'_j)$ and $f'_{l+1,i}(a_l) = a_l$. So $f'_{j,i}(M'_j) = f'_{l+1,i} \circ f'_{j,l+1}(M'_j)$ implies that $a_l \notin f'_{j,i}(M'_j)$.

Notice that for every $j < i$, $\check{M}$ is a (μ, μ^+)-limit over both M'_j and $f'_{j,i}(M'_j)$. Thus by the uniqueness of (μ, μ^+)-limit models, we can extend $f'_{j,i}$ to an automorphism of $\check{M}$, denoted by $\check{f}_{j,i}$. This completes the limit stage of the construction.

The construction is enough: Let M'_α and $\langle f_{i,\alpha} \mid i \leq \alpha\rangle$ be a direct limit of $(\langle M'_i \mid i < \alpha\rangle, \langle f_{j,i} \mid j \leq i < \alpha\rangle)$. By Subclaim II.6.14 we may assume that $\bigcup_{i<\alpha} M_i \prec_{\mathcal{K}} M'_\alpha$. It is routine to verify that $(\langle f_{i,\alpha}(M'_i) \mid i < \alpha\rangle, \bar{a})$ is a $<^b_{\mu,\alpha}$-extension of $(\bar{M}, \bar{a})$.

If $\bigcup_{i<\alpha} M_i$ is an amalgamation base we can find a $\mathcal{K}$-mapping as in the limit stage to choose $\bigcup_{i<\alpha} f'(M'_i) \prec_{\mathcal{K}} \check{M}$.

$\dashv$

Remark II.6.15. Notice that the extension $(\bar{M}', \bar{a})$ in Theorem II.6.10 is not continuous. Continuity of towers will be desired in the proof of the uniqueness of limit models. Taking an arbitrary $<^b$-extension will not give us a continuous tower. In fact, at this point, it is not apparent that any continuous extensions exist. However, in Section 2.9 we will show that reduced towers are continuous and reduced towers are dense. Thereby, allowing us to take continuous extensions.

2.7 $<^c_{\mu,\alpha}$ Extension Property for ${}^+\mathcal{K}^*_{\mu,\alpha}$

Unfortunately, it seems that working with the relatively simple $\mathcal{K}^*_{\mu,\alpha}$ towers is not sufficient to carry out the proof for the uniqueness of limit models. Shelah and Villaveces have idenitified a more elaborate tower. The extension proprerty for these towers is also missing from [ShVi]. We provide a partial solution to this extension property, analagous to the solution for $\mathcal{K}^*_{\mu,\alpha}$ in the previous section. In fact, we will have to further adjust our definition of towers to scattered towers in the following section. We introduce the scaled down towers of Sections 2.6

and 2.7 to break down the proof of the desired extension property into more manageable constructions.

We augment our towers with a dependence relation. The following variant of the first-order notion of splitting is often used in AECs. Most results relying on this notion are proved under the assumption of categoricity. Just recently progress has been made by considering μ-splitting in Galois-stable AECs see Chapter III.

Definition II.7.1. Let μ be a cardinal with $\mu < \lambda$. For $M \in \mathcal{K}^{am}$ and $p \in$ ga-S(M), we say that p *μ-splits over* N iff $N \prec_{\mathcal{K}} M$ and there exist $N_1, N_2 \in \mathcal{K}_\mu$ and a $\prec_{\mathcal{K}}$-mapping $h : N_1 \cong N_2$ such that

(1) $h(p \restriction N_1) \neq p \restriction N_2$,

(2) $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$ and

(3) $h \restriction N = id_N$.

Shelah and Villaveces draw a connection between categoricity and superstability-like properties by showing that under the assumption of categoricity there are no long splitting chains:

Theorem II.7.2 (Theorem 2.2.1 from [ShVi]). *Under Assumption II.1.1.(1) through II.1.1.(4), suppose that*

(1) $\langle M_i \mid i \leq \sigma\rangle$ *is* $\prec_{\mathcal{K}}$*-increasing and continuous,*

(2) for all $i \leq \sigma$, $M_i \in \mathcal{K}^{am}_\mu$,

(3) for all $i < \sigma$, M_{i+1} *is universal over* M_i

(4) $\mathrm{cf}(\sigma) = \sigma \leq \mu^+ \leq \lambda$ *and*

(5) $p \in$ ga-S(M_σ).

Then there exists $i < \sigma$ such that p does not μ-split over M_i.

Implicit in their proof of Theorem II.7.2 (case (a) in Theorem 2.2.1 from [ShVi]) is a statement which in the superstable first order case is an implication of $\kappa(T)$ being finite. This theorem is crucial for proving the uniqueness of limit models and its power may be exploited in the future to define a notion of forking (see Section **??**).

Theorem II.7.3. *Under Assumption II.1.1.(1) through II.1.1.(4), suppose that*

(1) $\langle M_i \mid i \leq \sigma \rangle$ *is* $\prec_{\mathcal{K}}$*-increasing and continuous,*

(2) for all $i \leq \sigma$, $M_i \in \mathcal{K}^{am}_{\mu}$,

(3) for all $i < \sigma$, M_{i+1} *is universal over* M_i,

(4) $\mathrm{cf}(\sigma) = \sigma \leq \mu^+ \leq \lambda$,

(5) $p \in \text{ga-S}(M_\sigma)$ *and*

(6) $p \restriction M_i$ *does not* μ*-split over* M_0 *for all* $i < \sigma$.

Then p does not μ-split over M_0.

? Note to Baldwin, Blum, Cummings and Schimmerling: The proof of Theorem II.7.2 in [ShVi] is surprisingly clear and well-written. There are 3 cases for the proof. Case (a) is exactly Theorem II.7.3 (although they do not state this as a separate result and it does not follow from the statement of Theorem II.7.2). But, there is nothing to change in the what is written to get the proof of Theorem II.7.3. In case you would still like me to include an exposition of the proof, let me know and I'll add it. ?!

Remark II.7.4. The proofs of Theorem II.7.2 and Theorem II.7.3 utilize the full power of the categoricity assumption. In particular, Shelah and Villaveces use the fact that every model can be embedded into a reduct of an Ehrenfreucht-Monstowski model. It is open as to whether or not similar theorems can be proven under the assumption of Galois-stability in every cardinality (Galois-superstablity).

We derive the extension property for non-splitting types (Theorem II.7.5). This result does not rely on the categoricity assumption. We will use it to find extensions of towers, but it is also useful for developing a stability theory for tame abstract elementary classes in Chapter III.

Theorem II.7.5 (Extension of non-splitting types). *Let $\check{M}$ be a (μ, μ^+)-limit containing $\bar{a} \bigcup M$. Suppose that $M \in \mathcal{K}_\mu$ is universal over N and* ga-tp$(a/M, \check{M})$ *does not μ-split over N.*

Let M' be an extension of M in $\mathcal{K}^{am}_\mu$. Then there exists a $\prec_{\mathcal{K}}$-mapping f such that $f : M' \to \check{M}$, $f \restriction M = id_M$ and ga-tp$(a/f(M'))$ *does not μ-split over N. Alternatively we can find $h \in \mathrm{Aut}_M(\check{M})$ such that $h : M' \to \check{M}$ and* ga-tp$(h(a)/M')$ *does not μ-split over N.*

Proof. We first prove the alternatively clause. Notice that $\check{M}$ is universal over M. So we may assume that $\check{M}$ contains M'. Since M is universal over N, there exists a $\prec_{\mathcal{K}}$ mapping $h' : M' \to M$ with $h' \restriction N = id_N$. By Proposition II.2.32, we can extend h' to an automorphism h of $\check{M}$. By invariance, ga-tp$(h^{-1}(a)/M'')$ does not μ-split over N.

Subclaim II.7.6. ga-tp$(h^{-1}(a)/M) =$ ga-tp(a/M).

Proof. We will use the notion of μ-splitting to prove this subclaim. So let us rename the models in such a way that our application of the definition μ-

splitting will become transparent. Let $N_1 := h^{-1}(M)$ and $N_2 = M$. Let $p :=$ ga-tp$(h^{-1}(a)/h^{-1}(M))$. Consider the mapping $h : N_1 \cong N_2$. Since p does not μ-split over N, $h(p \restriction N_1) = p \restriction N_2$. Let us calculate this

$$h(p \restriction N_1) = \text{ga-tp}(h(h^{-1}(a))/h(h^{-1}(M))) = \text{ga-tp}(a/M).$$

While,

$$p \restriction N_2 = \text{ga-tp}(h^{-1}(a)/M).$$

Thus ga-tp$(h^{-1}(a)/M) =$ ga-tp(a/M) as required. $\dashv$

From the subclaim, we can find a $\prec_{\mathcal{K}}$-mapping g and a model $M^* \prec_{\mathcal{K}} \check{M}$ such that $g : M'' \to M^*$, $g \restriction M = id_M$ and $g \circ h^{-1}(a) = a$. Notice that ga-tp$(a/g(M''), \check{M})$ does not μ-split over M. $\dashv$

Now we incorporate μ-splitting into our definition of towers.

Definition II.7.7.

$$^+\mathcal{K}^*_{\mu,\alpha} := \left\{ (\bar{M}, \bar{a}, \bar{N}) \;\middle|\; \begin{array}{l} (\bar{M}, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}; \\ \bar{N} = \langle N_i \mid i+1 < \alpha \rangle; \\ \text{for every } i+1 < \alpha, N_i \prec_{\mathcal{K}} M_i, ; \\ M_i \text{ is universal over } N_i \text{ and;} \\ \text{ga-tp}(a_i, M_i, M_{i+1}) \text{ does not } \mu\text{-split over } N_i. \end{array} \right\}$$

Similar to the case of $\mathcal{K}^*_{\mu,\alpha}$ we define an ordering,

Definition II.7.8. For $(\bar{M}, \bar{a}, \bar{N})$ and $(\bar{M}', \bar{a}', \bar{N}') \in {}^+\mathcal{K}^*_{\mu,\alpha}$, we say $(\bar{M}, \bar{a}, \bar{N}) <^c_{\mu,\alpha}$ $(\bar{M}', \bar{a}', \bar{N}')$ iff

(1) $(\bar{M}, \bar{a}) <^b_{\mu,\alpha} (\bar{M}', \bar{a}')$

(2) $\bar{N} = \bar{N}'$ and

(3) for every $i < \alpha$, ga-tp$(a_i/M'_i, M'_{i+1})$ does not μ-split over N_i.

Remark II.7.9. Notice that in Definition II.7.8, condition (3) follows from (2). We list it as a separate condition to emphasize the role of μ-splitting.

Notation II.7.10. We say that $(\bar{M}, \bar{a}, \bar{N})$ is *nice* iff when i is a limit ordinal $\bigcup_{j<i} M_j$ is an amalgamation base.

The following theorem is a partial solution to a problem from [ShVi]:

Theorem II.7.11 (The $<^c_{\mu,\alpha}$-extension property for nice towers). *If $(\bar{M}, \bar{a}, \bar{N}) \in {}^+\mathcal{K}^*_{\mu,\alpha}$ is nice, then there exists a nice $(\bar{M}', \bar{a}, \bar{N}') \in {}^+\mathcal{K}^*_{\mu,\alpha}$ such that $(\bar{M}, \bar{a}, \bar{N}) <^c_{\mu,\alpha} (\bar{M}', \bar{a}, \bar{N}')$. Moreover if $\bigcup_{i<\alpha} M_i$ is an amalgamation base such that $\bigcup_{i<\alpha} M_i \prec_{\mathcal{K}} \check{M}$ for some (μ, μ^+)-limit, $\check{M}$, then we can find $(\bar{M}', \bar{a}', \bar{N}')$ such that $\bigcup_{i<\alpha} M'_i \prec_{\mathcal{K}} \check{M}$.*

Proof. Let μ be a cardinal and α a limit ordinal such that $\alpha < \mu^+ \leq \lambda$. Let $(\bar{M}, \bar{a}, \bar{N}) \in {}^+\mathcal{K}^*_{\mu,\alpha}$ be given. Denote by M_α a model in $\mathcal{K}^{am}_\mu$ extending $\bigcup_{i<\alpha} M_i$. Fix $\check{M}$ to be a (μ, μ^+)-limit model over M_α.

Similar to the proof of Theorem II.6.10, we will define by induction on $i < \alpha$ a sequence of models $\langle M'_i \mid i < \alpha\rangle$ and sequences of $\prec_{\mathcal{K}}$-mappings, $\langle f'_{j,i} \mid j < i < \alpha\rangle$ and $\langle \check{f}_{j,i} \mid j < i < \alpha\rangle$ such that for $i \leq \alpha$:

(1) $(\langle f'_{j,i}(M'_j) \mid j \leq i\rangle, \bar{a} \restriction i, \bar{N} \restriction i)$ is a $<^c_{\mu,i}$-extension of $(\bar{M}, \bar{a}, \bar{N}) \restriction i$,

(2) $(\langle M'_j \mid j < i\rangle, \langle f'_{j,i} \mid j \leq i\rangle)$ forms a directed system,

(3) M'_i is universal over M_i,

(4) M'_{i+1} is universal over $f'_{i,i+1}(M'_i)$,

(5) $f'_{j,i} \restriction M_j = id_{M_j}$,

(6) $M'_i \prec_{\mathcal{K}} \check{M}$,

(7) $f'_{j,i}$ can be extended to an automorphism of $\check{M}$, $\check{f}_{j,i}$, for $j \leq i$ and

(8) $(\langle \check{M} \mid j < i\rangle, \langle \check{f}_{k,j} \mid k \leq j < i\rangle)$ forms a directed system.

The construction is enough: We can take M'_α and $\langle f'_{i,\alpha} \mid i < \alpha\rangle$ to be a direct limit of $(\langle M'_i \mid i < \alpha\rangle, \langle f'_{j,i} \mid j \leq i < \alpha\rangle)$. Since $f'_{j,i} \restriction M_j = id_{M_j}$, for every $j \leq i < \alpha$, we may assume that $f'_{i,\alpha} \restriction M_i = id_{M_i}$ for every $i < \alpha$. Notice that $(\langle f'_{i,\alpha}(M'_i) \mid i < \alpha\rangle, \bar{a})$ is a $<^c_{\mu,\alpha}$-extension of $(\bar{M}, \bar{a})$. For the moreover part, simply continue the construction one more step for $i = \alpha$.

The construction is possible:

$i = 0$: Since M_0 is an amalgamation base, we can find $M''_0 \in \mathcal{K}^*_\mu$ (a first approximation of the desired M'_0) such that M''_0 is universal over M_0. By Theorem II.7.5, we may assume that ga-tp(a_0/M''_0) does not μ-split over N_0 and $M''_0 \prec_{\mathcal{K}} \check{M}$. Since $a_0 \notin M_0$ and ga-tp(a_0/M_0) does not μ-split over N_0, we know that $a_0 \notin M''_0$. But, we might have that for some $l > 0$, $a_l \in M''_0$. We use weak disjoint amalgamation to avoid $\{a_l \mid 0 < l < \alpha\}$. By the Downward Löweneim-Skolem Axiom for AECs (Axiom 6) we can choose $M^2 \in \mathcal{K}_\mu$ such that $M''_0, M_1 \prec_{\mathcal{K}} M^2 \prec_{\mathcal{K}} \check{M}$.

By Corollary II.5.3 (applied to M_1, M_α, M^2 and $\langle a_l \mid 0 < l < \alpha\rangle$), we can find a $\prec_{\mathcal{K}}$-mapping f such that

- $f : M^2 \to \check{M}$
- $f \restriction M_1 = id_{M_1}$
- $f(M^2) \cap \{a_l \mid 0 < l < \alpha\} = \emptyset$

Define $M'_0 := f(M''_0)$. Notice that $a_0 \notin M'_0$ because $a_0 \notin M''_0$ and $f(a_0) = a_0$. Clearly $M'_0 \cap \{a_l \mid 0 \leq l < \alpha\} = \emptyset$, since $M''_0 \prec_{\mathcal{K}} M^2$ and $f(M^2) \cap$

$\{a_l \mid 0 < l < \alpha\} = \emptyset$. We need only verify that ga-tp(a_0/M_0') does not μ-split over N_0. By invariance, ga-tp(a_0/M_0'') does not μ-split over N_0 implies that ga-tp$(f(a_0)/f(M_0''))$ does not μ-split over N_0. But recall $f(a_0) = a_0$ and $f(M_0'') = M_0'$. Thus ga-tp(a_0/M_0') does not μ-split over N_0.

Set $\check{f}_{0,0} := id_{\check{M}}$ and $f'_{0,0} := id_{M_0'}$.

$i = j + 1$: Suppose that we have completed the construction for all $k \leq j$. Since $M_j', M_{j+1} \prec_{\mathcal{K}} \check{M}$, we can apply the Downward-Löwenheim Axiom for AECs to find M_{j+1}''' (a first approximation to M_{j+1}') a model of cardinality μ extending both M_j' and M_{j+1}. WLOG by Subclaim II.6.12 we may assume that M_{j+1}''' is a limit model of cardinality μ and M_{j+1}''' is universal over M_{j+1} and M_j'. By Theorem II.7.5, we can find a $\prec_{\mathcal{K}}$ mapping $f : M_{j+1}''' \to \check{M}$ such that $f \restriction M_{j+1} = id_{M_{j+1}}$ and ga-tp$(a_{j+1}/f(M_{j+1}'''))$ does not μ-split over N_{j+1}. Set $M_{j+1}'' := f(M_{j+1}''')$.

Subclaim II.7.12. $a_{j+1} \notin M_{j+1}''$

Proof. Suppose that $a_{j+1} \in M_{j+1}''$. Since M_{j+1}' is universal over N_{j+1}, there exists a $\prec_{\mathcal{K}}$-mapping, $g : M_{j+1}'' \to M_{j+1}'$ such that $g \restriction N_{j+1} = id_{N_{j+1}}$. Since ga-tp$(a_{j+1}/M_{j+1}'')$ does not μ-split over N_{j+1}, we have that

$$\text{ga-tp}(a_{j+1}/g(M_{j+1}'') = \text{ga-tp}(g(a_{j+1})/g(M_{j+1}'')).$$

Notice that because $g(a_{j+1}) \in g(M_{j+1}'')$, we have that $a_{j+1} = g(a_{j+1})$. Thus $a_{j+1} \in g(M_{j+1}'') \prec_{\mathcal{K}} M_{j+1}$. This contradicts the definition of towers: $a_{j+1} \notin M_{j+1}$.

$\dashv$

M_{j+1}'' may serve us well if it does not contain any a_l for $j + 1 \leq l < \alpha$, but this is not guaranteed. So we need to make an adjustment. Let M^2 be a model of cardinality μ such that $M_{j+2}, M''_{j+1} \prec_{\mathcal{K}} M^2 \prec_{\mathcal{K}} \check{M}$. Notice that $\check{M}$ is universal over M_{j+2}. Thus we can apply Corollary II.5.3 to M_{j+2}, M_α, M^2 and $\langle a_l \mid j+2 \leq l < \alpha\rangle$. This yields a $\prec_{\mathcal{K}}$-mapping h such that

- $h : M^2 \to \check{M}$
- $h \restriction M_{j+2} = id_{M_{j+2}}$ and
- $h(M^2) \cap \{a_l \mid j+2 \leq l < \alpha\} = \emptyset$.

Set $M'_{j+1} := h(M''_{j+1})$. Notice that by invariance, ga-tp(a_{j+1}/M''_{j+1}) does not μ-split over N_{j+1} implies that ga-tp$(h(a_{j+1})/h(M''_{j+1}))$ does not μ-split over $h(N_{j+1})$. Recalling that $h \restriction M_{j+2} = id_{M_{j+2}}$ we have that ga-tp(a_{j+1}/M''_{j+1}) does not μ-split over N_{j+1}. We need to verify that $a_{j+1} \notin M'_{j+1}$. This holds because $a_{j+1} \notin M''_{j+1}$ and $h(a_{j+1}) = a_{j+1}$.

Set $f'_{j+1,j+1} = id_{M_{j+1,j+1}}$ and $\check{f}_{j+1,j+1} = id_{\check{M}}$ and $f'_{j,j+1} := h \circ f \restriction M'_j$. Since $\check{M}$ is a (μ, μ^+)-limit over both M'_j and $f'_{j,j+1}(M'_j)$, we can extend $f'_{j,j+1}$ to an automorphism of $\check{M}$, denoted by $\check{f}_{j,j+1}$.

To guarantee that we have a directed system, for $k < j$, define $f'_{k,j+1} := f'_{j,j+1} \circ f'_{k,j}$ and $\check{f}_{k,j+1} := \check{f}_{j,j+1} \circ \check{f}_{k,j}$.

i is a limit ordinal: Suppose that $(\langle M'_j \mid j < i\rangle, \langle f'_{k,j} \mid k \leq j < i\rangle)$ and $(\langle \check{M} \mid j < i\rangle, \langle \check{f}_{k,j} \mid k \leq j < i\rangle)$ have been defined. Since they are both directed systems, we can take direct limits. By niceness we can apply Claim II.6.13, so that we may assume that $(M^*_i, \langle f^*_{j,i} \mid j < i\rangle)$ and $(\check{M}, \langle \check{f}^*_{j,i} \mid j < i\rangle)$ are the respective direct limits such that $M^*_i \prec_{\mathcal{K}} \check{M}$ and $\bigcup_{j<i} M_j \prec_{\mathcal{K}} M^*_i$. By Condition (4) of the construction, notice that M^*_i is a (μ, i)-limit model witnessed by $\langle f^*_{j,i}(M'_j) \mid j < i\rangle$. Hence M^*_i is an amalgamation base. Since M^*_i

and M_i both live inside of $\check{M}$, we can find $M_i''' \in \mathcal{K}^*_\mu$ which is universal over M_i and universal over M_i^*.

By Theorem II.7.5 we can find a $\prec_{\mathcal{K}}$-mapping $f : M_i''' \to \check{M}$ such that $f \restriction M_i = id_{M_i}$ and ga-tp$(a_i/f(M_i'''))$ does not μ-split over N_i. Set $M_i'' := f(M_i''')$. By a similar argument to Subclaim II.7.12, we can see that $a_i \notin M_i''$.

M_i'' may contain some a_l when $i \leq l < \alpha$. We need to make an adjustment using weak disjoint amalgamtion. Let M^2 be a model of cardinality μ such that $M_i'', M_{i+1} \prec_{\mathcal{K}} M^2 \prec_{\mathcal{K}} \check{M}$. By Corollary II.5.3 applied to M_i, M_α, M^2 and $\langle a_l \mid i < l < \alpha \rangle$ we can find $h : M_i'' \to \check{M}$ such that $h \restriction M_{i+1} = id_{M_{i+1}}$ and $h(M^2) \cap \{a_l \mid i < l < \alpha\} = \emptyset$.

Set $M_i' := h(M_i'')$. We need to verify that $a_i \notin M_i'$ and ga-tp(a_i/M_i') does not μ-split over N_i. Since $a_i \notin M_i''$ and $h(a_i) = a_i$, we have that $a_i \notin h(M_i'') = M_i'$. By invariance of non-splitting, ga-tp(a_i/M_i'') not μ-splitting over N_i implies that ga-tp$(h(a_i)/h(M_i''))$ does not μ-split over $h(N_i)$. Recalling our definition of h and M_i'. This yields ga-tp(a_i/M_i') does not μ-split over N_i.

Set $f'_{i,i} := id_{M_i,i}$, $\check{f}_{i,i} := id_{\check{M}}$ and for $j < i$, $f'_{j,i} := h \circ f \circ f^*_{j,i}$.

Notice that for every $j < i$, $\check{M}$ is a (μ, μ^+)-limit over both M_j' and $f'_{j,i}(M_j')$. Thus by the uniqueness of (μ, μ^+)-limit models, we can extend $f'_{j,i}$ to an automorphism of $\check{M}$, denoted by $\check{f}_{j,i}$.

$\dashv$

2.8 Extension Property for Scattered Towers

We now make the final modification to the towers and prove an extension theorem for these scattered towers. Let's recall the general strategy for proving the uniqueness of limit models. Our goal is to construct an array of models $\langle M^i_j \mid j \leq \theta_1, i \leq \theta_2\rangle$ of width θ_1 and height θ_2 such that the union will be simultaneously a (μ,θ_1)-limit model (witnessed by $\langle M^{\theta_2}_j \mid j < \theta_1\rangle$) and a (μ,θ_2)-limit model (witnessed by $\langle M^i_{\theta_1} \mid i < \theta_2\rangle$). In spirit our construction will behave this way. However, such a construction is too much to hope for because:

(1) We would like to focus on towers $(\bar{M},\bar{a},\bar{N}) \in {}^+\mathcal{K}^*_{\mu,\alpha}$ such that

$$(*) \quad M_{i+1} \text{ is universal over } M_i.$$

That way, $\bigcup_{i<\alpha} M_i$ would be a (μ,α)-limit model. While these towers are easy to construct individually, if we were to construct a $<^c_{\mu,\alpha}$-increasing and continuous chain of such towers, $\langle(\bar{M},\bar{a},\bar{N})^\beta \mid \beta < \alpha\rangle$, we would not necessarily know at limit stages, $\beta < \alpha$, that the tower $(\bar{M},\bar{a},\bar{N})^\beta$ satisfies $(*)$.

(2) While our ordering on towers is enough to get that $M^{\theta_2}_i$ is a (μ,θ_2)-limit for $i < \theta_1$ (witnessed by $\langle M^j_i \mid j < \theta_2\rangle$), we cannot say anything about the model $M^{\theta_2}_{\theta_1}$. Unfortunately it is not reasonable to "fix" our definition of ordering to guarantee that $M^{\theta_2}_{\theta_1}$ is a limit model, since we would then be unable (at least we see no way of doing it directly) to prove the extension property for towers.

In Sections 2.9 and 2.10, we address problem (1) by identifying some properties of towers (full and reduced) that guarantee that the top of the tower ($M^{\sigma_2}_{\sigma_1}$) is in fact a (μ,σ_1)-limit model.

To remedy (2) we define scattered towers. Since we know that $M^{\theta_2}_i$ is a (μ,θ_2)-limit for $i < \theta_1$ (witnessed by $\langle M^j_i \mid j < \theta_2\rangle$), the idea is to construct a very wide array of towers (of width μ^+) and then focus in on some $\alpha < \mu^+$ of cofinality θ_1.

Then $M_\alpha^{\theta_2}$ won't be in the last column of the array, so the ordering will guarantee us that $M_\alpha^{\theta_2}$ is a (μ, θ_2)-limit (witnessed by $\langle M_\alpha^j \mid j < \theta_2 \rangle$). However, we have not proved an extension property for towers of width μ^+. Our arguments won't generalize to $\mathcal{K}_{\mu,\mu^+}$ because Theorem II.5.1 (Weak Disjoint Amalgamation) isn't strong enough since we would have μ^+ many elements to avoid ($\{a_i \mid i < \mu^+\}$). So we will construct the tower in $\mathcal{K}_{\mu,\mu^+}$ in μ^+-many stages by shorter towers (in $\mathcal{K}^*_{\mu,\alpha}$ for $\alpha < \mu^+$). To do this we introduce the notion of scattered towers, which will allow us to extend a tower in $\mathcal{K}^*_{\mu,\alpha}$ to a longer tower in $\mathcal{K}^*_{\mu,\beta}$ when $\alpha < \beta < \mu^+$.

Notation II.8.1. Let α be an ordinal. We say that $\mathfrak{U} \subseteq \mathcal{P}(\alpha)$ is a *set of disjoint intervals of* α *of which one contains* 0 provided that

- $0 \in \bigcup \mathfrak{U}$,
- for $u_1 \neq u_2 \in \mathfrak{U}$, $u_1 \cap u_2 = \emptyset$ and
- for $u \in \mathfrak{U}$, if $\beta_1 < \beta_2 \in u$, then for every γ with $\beta_1 < \gamma < \beta_2$, we have $\gamma \in u$.

Since we will not be looking at any other sets of intervals, we abbreviate a *set of disjoint intervals of* α *of which one contains* 0 as a *set of intervals.*

Definition II.8.2 (Definition 3.3.1 of [ShVi]). For $\mathfrak{U}$ a set of intervals of ordinals $< \mu^+$, let

$$
{}^+\mathcal{K}^*_{\mu,\mathfrak{U}} := \left\{ (\bar{M}, \bar{a}, \bar{N}) \;\middle|\; \begin{array}{l} \bar{M} = \langle M_i \mid i \in u \text{ for some interval } u \in \mathfrak{U}\rangle; \\ \bar{M} \text{ is } \prec_{\mathcal{K}} \text{ increasing, but not} \\ \text{necessarily continuous;} \\ a_i \in M_{i+1} \backslash M_i \text{ when } i, i+1 \in \bigcup \mathfrak{U}; \\ \bar{N} = \langle N_i \mid i \in \bigcup \mathfrak{U}\rangle; \\ M_i \text{ is universal over } N_i \text{ when } i, i+1 \in \bigcup \mathfrak{U} \text{ and} \\ \text{ga-tp}(a_i, M_i, M_{i+1}) \text{ does not } \mu-\text{split over } N_i \end{array} \right\}
$$

Notice that these *scattered towers* are in some sense subtowers of the towers ${}^+\mathcal{K}^*_{\mu,\alpha}$. Hence we can consider the restriction of $<^c_{\mu,\alpha}$ to the class ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}}$:

Definition II.8.3 (Definition 3.3.2 of [ShVi]). Let $(\bar{M}^l, \bar{a}^l, \bar{N}^l) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}}$ for $l = 1, 2$. $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \leq^c (\bar{M}^2, \bar{a}^2, \bar{N}^2)$ iff for $i \in \bigcup \mathfrak{U}$,

(1) $M^1_i \preceq_{\mathcal{K}} M^2_i$, $a^1_i = a^2_i$ and $N^1_i = N^2_i$ and

(2) if $M^1_i \neq M^2_i$, then M^2_i is universal over M^1_i.

We say that $(\bar{M}^1, \bar{a}^1, \bar{N}^1) <^c (\bar{M}^2, \bar{a}^2, \bar{N}^2)$ provided that for every $i \in \bigcup \mathfrak{U}$, $M^1_i \neq M^2_i$.

Actually we can extend the ordering to compare towers from classes ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}_1}$ and ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}_2}$ when $\mathfrak{U}_2$ is an interval-extension of $\mathfrak{U}_1$. By interval-extension we mean:

Definition II.8.4. $\mathfrak{U}_2$ is an *interval-extension of* $\mathfrak{U}_1$ iff for every $u_1 \in \mathfrak{U}_1$, there is $u_2 \in \mathfrak{U}_2$ such that $u_1 \subseteq u_2$. We write $\mathfrak{U}^1 \subset_{int} \mathfrak{U}^2$ when $\mathfrak{U}^2$ is an interval extension of $\mathfrak{U}^1$.

Definition II.8.5. Let $\mathfrak{U}^2$ be an interval extension of $\mathfrak{U}^1$. Let $(\bar{M}^l, \bar{a}^l, \bar{N}^l) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}_l}$ for $l = 1, 2$. $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \leq^c (\bar{M}^2, \bar{a}^2, \bar{N}^2)$ iff for $i \in \bigcup \mathfrak{U}_1$,

(1) $M_i^1 \preceq_{\mathcal{K}} M_i^2$, $a_i^1 = a_i^2$ and $N_i^1 = N_i^2$ and

(2) if $M_i^1 \neq M_i^2$, then M_i^2 is universal over M_i^1.

Now we can generalize the notion of niceness and prove an extension property for the class of all scattered towers.

Definition II.8.6. A scattered tower $(\bar{M}, \bar{a}, \bar{N}) \in {}^{+}\mathcal{K}^*_{\mu,\mathfrak{U}}$ is said to be *nice* provided that whenever a limit ordinal i is a limit of some sequence of elements from $\bigcup \mathfrak{U}$, then $\bigcup_{j \in \bigcup \mathfrak{U},\, j<i} M_j$ is an amalgamation base.

Theorem II.8.7 ($<^c$-Extension Property for Nice Scattered Towers). *Let $\mathfrak{U}^1$ and $\mathfrak{U}^2$ be sets of intervals of ordinals $< \mu^+$ such that $\mathfrak{U}^2$ is an interval extension of $\mathfrak{U}^1$. Let $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \in {}^{+}\mathcal{K}^*_{\mu,\mathfrak{U}^1}$ be a nice scattered tower. There exists a nice scattered tower $(\bar{M}^2, \bar{a}^2, \bar{N}^2) \in {}^{+}\mathcal{K}^*_{\mu,\mathfrak{U}^2}$ such that $(\bar{M}^1, \bar{a}^1, \bar{N}^1) <^c (\bar{M}^2, \bar{a}^2, \bar{N}^2)$.*

Moreover, if $\bigcup_{i \in \bigcup \mathfrak{U}} M_i$ is an amalgamation base and $\bigcup_{i \in \bigcup \mathfrak{U}} M_i \prec_{\mathcal{K}} \check{M}$ for some (μ, μ^+)-limit $\check{M}$, then we can find $(\bar{M}', \bar{a}', \bar{N}')$ such that $\bigcup_{i \in \bigcup \mathfrak{U}} M_i \prec_{\mathcal{K}} \check{M}$.

Proof. WLOG we can rewrite $\mathfrak{U}^2$ as a collection of disjoint intervals such that for every $u^2 \in \mathfrak{U}^2$, there exists at most one $u^1 \in \mathfrak{U}^1$ such that $u^1 \subseteq u^2$. Let us enumerate $\mathfrak{U}^1$ as $\langle u_t^1 \mid t \in \alpha^1 \rangle$ in increasing order (in other words when $t < t' \in \alpha^1$ we have that $\max(u_t^1) < \min(u_{t'}^1)$.)

For bookkeeping purposes, we will enumerate $\mathfrak{U}^2$ as $\langle u_t^2 \mid t \in \alpha^1 \rangle$ as

$$u_t^2 = \begin{cases} \left\{ i \in \bigcup \mathfrak{U}^2 \mid \min\{u_t^1\} \leq i < \min\{u_{t+1}^1\} \right\} & \text{if } t+1 < \alpha^1 \\ \left\{ i \in \bigcup \mathfrak{U}^2 \mid \min\{u_t^1\} \leq i \right\} & \text{otherwise} \end{cases}$$

Remark II.8.8. The second part of the definition of u_t^2 is used only to define $u_{\alpha^1}^2$ when α^1 is a successor ordinal.

Since $0 \in \bigcup \mathfrak{U}^1$, this enumeration of $\mathfrak{U}^2$ can be carried out.

Given $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}^1}$ a nice tower, we will find a $<^c$-extension in ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}^2}$ by using direct limits inside a (μ,μ^+)-limit model as we have done in the proofs of Theorem II.6.10 and Theorem II.7.11. As before, fix $\check{M}$ a (μ,μ^+)-limit model containing $\bigcup_{i\in\bigcup\mathfrak{U}^1} M^1_i$. We will define approximations to a tower in ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}^2}$ with towers in ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}^2_t}$ extending towers in ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}^1_t}$ where $\mathfrak{U}^l_t = \{u^l_s \mid s \leq t\}$ for $l = 1, 2$.

These partial extensions will be defined by constructing sequences of models $\langle M^2_i \mid i \in \bigcup\mathfrak{U}^2\rangle$ and $\langle N^2_i \mid i, i+1 \in \bigcup\mathfrak{U}^2\rangle$, a sequence of elements $\langle a^2_i \mid i, i+1 \in \bigcup\mathfrak{U}^2\rangle$ and $\prec_{\mathcal{K}}$-mappings $\{f_{s,t} \mid s \leq t < \alpha^1\}$ (or $\{f_{s,t} \mid s \leq t \leq \alpha^1\}$ for α^1 a sucessor) satisfying

(1) $(\langle f_{s,t}(M^2_i) \mid i \in u^2_s \text{ and } s \leq t\rangle, \bar{a}^t, \bar{N}^t)$ is a $<^c_{\mu,\mathfrak{U}^1_t}$-extension of $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \restriction \mathfrak{U}^1_t$ where $\bar{a}^t = \langle a^2_i \mid i, i+1 \in \mathfrak{U}^2_t\rangle$ and $\bar{N}^t = \langle N^2_i \mid i, i+1 \in \mathfrak{U}^2_t\rangle$,

(2) $(\langle M^s \mid s \leq t\rangle, \langle f_{s,t} \mid s \leq t\rangle)$ forms a directed system where $M^s = \bigcup_{i\in u^2_s} M^2_i$.

(3) M^2_i is universal over M^1_i for all $i \in \bigcup\mathfrak{U}^2_t$,

(4) M^2_j is universal over $f_{s,t}(M^2_i)$ for every $i < j$ and $s \leq t$ such that $i \in u^2_s$ and $j \in u^2_t$ (consequently, M^{t+1} is universal over $f_{t,t+1}(M^t)$),

(5) $f_{s,t} \restriction M^1_j = id_{M^1_j}$ for all $j \in u^2_s$,

(6) $M^2_i \prec_{\mathcal{K}} \check{M}$,

(7) $f_{s,t}$ can be extended to an automorphism of $\check{M}$, $\check{f}_{s,t}$, for $s \leq t < \alpha^1$ and

(8) $(\langle \check{M} \mid s \leq t\rangle, \langle \check{f}_{s,t} \mid s \leq t\rangle)$ forms a directed system.

The construction is enough:

Let $\alpha := \alpha^1$ if α^1 is a limit, otherwise $\alpha := \alpha^1 + 1$. We can take M'_α and $\langle f_{t,\alpha} \mid t \leq \alpha\rangle$ to be a direct limit of $(\langle M^t \mid t < \alpha\rangle, \langle f_{s,t} \mid s \leq t < \alpha\rangle)$. Since $f_{s,t} \restriction$

$M_i^1 = id_{M_i^1}$, for every $i \in u_s^2$, we may assume that $f_{t,\alpha} \restriction M^t = id_{M^t}$ for every $t < \alpha$. Notice that $(\langle f_{t,\alpha}(M_i') \mid i \in u_t^2,\ t < \alpha \rangle, \langle a_i^2 \mid i \in \bigcup \mathfrak{U}^2 \rangle, \langle N_i^2 \mid i \in \bigcup \mathfrak{U}^2 \rangle)$ is a $<^c_{\mu,\alpha}$-extension of $(\bar{M}, \bar{a}, \bar{N})^1$. For the moreover part, simply continue the construction one more limit step.

The construction:

$t = 0$: First notice that by Theorem II.7.11, we can find $\langle M_i' \mid i \in u_0^1 \rangle$ such that $(\bar{M}', \bar{a}^1 \restriction u_0^1, \bar{N}^1 \restriction u_0^1)$ is a $<^c_{\mathfrak{U}_0^1}$-extension of $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \restriction \mathfrak{U}_0^1$ and $\bar{M}'$ avoids $\bar{a}^1$ above u_0^1 (specifically $(\bigcup_{i \in u_0^1} M_i') \bigcap \{a^j \mid j \in \bigcup \mathfrak{U}^1 \setminus u_0^1\} = \emptyset$.) Moreover the proof of Theorem 7.10 gives us an extension such that $\bigcup_{i \in u_0^1} M_i'$ is a limit model.

We can choose $M^\dagger \in \mathcal{K}_\mu$ such that $\bigcup_{i \in u_0^1} M_i'$, $M^1_{\min\{u_1^1\}} \prec_{\mathcal{K}} M^\dagger \prec_{\mathcal{K}} \check{M}$ and $M^\dagger$ is a $(\mu, |u_0^2| + \aleph_0)$-limit over $\bigcup_{i \in u_0^1} M_i'$. This is possible since $\bigcup_{i \in u_0^1} M_i'$ is an amalgamation base. Let $\langle M_\gamma^\dagger \mid \gamma < |u_0^2| + \aleph_0 \rangle$ witness that $M^\dagger$ is a $(\mu, |u_0^2| + \aleph_0)$-limit over $\bigcup_{i \in u_0^1} M_i'$. Since limit models are amalgamation bases, we may choose $M_{\gamma+1}^\dagger$ to be a (μ, ω)-limit over $M_\gamma^\dagger$.

By weak disjoint amlagamation (Corollary II.5.3) applied to $(\bigcup_{i \in u_0^1} M_i^1, \bigcup_{i \in u_0^1} M_i', M^\dagger)$ and $\{a_j^1 \mid j \in \bigcup \mathfrak{U}^1 \setminus \mathfrak{U}_0^1\}$, there exists an automorphism g of $\check{M}$ such that

- $g \restriction \bigcup_{i \in u_0^1} M_i^1 = id_{\bigcup_{i \in u_0^1} M_i^1}$ and
- $g(M^\dagger) \cap \{a_j^1 \mid j \in \bigcup \mathfrak{U}^1 \setminus u_0^1\} = \emptyset$.

Denote by $\langle i_\gamma \mid \gamma \in \text{otp}(u_0^2 \setminus u_0^1) \rangle$ the increasing enumeration of $u_0^2 \setminus u_0^1$. Define

$$M_i^2 := \begin{cases} g(M_i') & \text{for } i \in u_0^1 \\ g(M_\gamma^\dagger) & \text{for } i = i_\gamma \in u_0^2 \setminus u_0^1 \end{cases}$$

Since $M^\dagger$ is an limit model witnessed by the $M_\gamma^\dagger$'s, we can choose $a_i \in M_{i+1}^2 \setminus M_i^2$ for all $i, i+1 \in u_0^2 \setminus u_0^1$. Since M_i^2 is a limit model for each $i, i+1 \in u_0^2 \setminus u_0^1$, we can apply Theorem II.7.2 to find $N_i^2 \prec_{\mathcal{K}} M_i^2$ such that ga-tp(a_i/M_i^2) does not μ-split over N_i^2 and M_i^2 is universal over N_i^2.

All that remains is to define $f_{0,0} := id_{\bigcup_{i\in u_0^1} M_i^1}$ and $\check{f}_{0,0} := id_{\check{M}}$.

$t = s+1$: By condition (4) of the construction, we have that $\bigcup_{i\in u_s^2} M_i^2$ is a limit model witnessed by $\langle f_{r,s}(M_i^2) \mid i \in u_r^2 \text{ and } r \leq s\rangle$. Thus $\bigcup_{i\in u_s^2} M_i^2$ is an amalgamation base. Now we can choose a model $M' \in \mathcal{K}_\mu$ such that $\bigcup_{i\in u_s^2} M_i^2$, $M^1_{\min\{u^1_{s+1}\}} \prec_{\mathcal{K}} M'$ and M'' is a $(\mu, |u^2_{s+1}| + \aleph_0)$-limit over $\bigcup_{i\in u_s^2} M_i^2$. By identical arguments to the successor case in Theorem II.7.11, we can find $\bar{M}' = \langle M'_i \mid i \in \mathfrak{U}^2_s \bigcup u^1_{s+1}\rangle$ and an automorphism h of $\check{M}$ such that

- $(\bar{M}', \bar{a}', \bar{N}')$ is a nice scattered tower, where $\bar{a}' = \langle a_i^2 \mid i \in \mathfrak{U}^2_s \bigcup u^1_{s+1}\rangle$ and $\bar{N}' = \langle N_i^2 \mid i \in \mathfrak{U}^2_s \bigcup u^1_{s+1}\rangle$
- $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \restriction \mathfrak{U}^1_{s+1} <^c (\bar{M}', \bar{a}', \bar{N}')$
- $\bigcup_{i\in \mathfrak{U}^2_s \bigcup u^1_{s+1}} M'_i \cap \{a^1_j \mid j \in \mathfrak{U}^1 \setminus \mathfrak{U}^1_{s+1}\} = \emptyset$.
- $h \restriction M'' : M'' \cong M'_{\min\{u^1_{s+1}\}}$ and
- $h \restriction M^1_{\min\{u^1_{s+1}\}} = id_{M^1_{\min\{u^1_{s+1}\}}}$.

Let $M^\dagger$ be a $(\mu, |u^2_{s+1}\backslash u^1_{s+1}| + \aleph_0)$-limit model over $\bigcup_{i\in \mathfrak{U}^2_s \bigcup u^1_{s+1}} M'_i$ such that such that $M^1_{\min\{u^2_{s+2}\}} \prec_{\mathcal{K}} M^\dagger \prec_{\mathcal{K}} \check{M}$. Let $\langle M^\dagger_\gamma \mid \gamma < |u^2_{s+1}\backslash u^1_{s+1}| + \aleph_0\rangle$ witness that $M^\dagger$ is a limit model. Since limit models are amalgamation bases, we may choose $M^\dagger_{\gamma+1}$ to be a (μ,ω)-limit over $M^\dagger_\gamma$.

Applying Corollary II.5.3 to $(\bigcup_{i\in u^1_{s+1}} M^1_i, \bigcup_{i\in \mathfrak{U}^2_s \bigcup u^1_{s+1}} M'_i, M^\dagger)$ and $\{a^1_j \mid j \in \bigcup \mathfrak{U}^1 \setminus \mathfrak{U}^1_{s+1}\}$, there exists an automorphism of $\check{M}$, g, such that

- $g \restriction \bigcup_{i\in u^1_{s+1}} M^1_i = id_{\bigcup_{i\in u^1_{s+1}} M^1_i}$ and
- $g(M^\dagger) \cap \{a^1_j \mid j \in \bigcup \mathfrak{U}^1 \setminus \mathfrak{U}^1_{s+1}\} = \emptyset$.

Denote by $\langle i_\gamma \mid \gamma \in \mathrm{otp}(u^2_{s+1}\backslash u^1_{s+1})\rangle$ the increasing enumeration of $u^2_{s+1}\backslash u^1_{s+1}$. Define

$$M^2_i := \begin{cases} g(M'_i) & \text{for } i \in u^1_{s+1} \\ g(M^\dagger_\gamma) & \text{for } i = i_\gamma \in u^2_{s+1}\backslash u^1_{s+1} \end{cases}$$

Since $M^\dagger$ is a limit model witnessed by the $M^\dagger_\gamma$'s, we can choose $a_i \in M^2_{i+1}\backslash M^2_i$ for all $i, i+1 \in u^2_{s+1}\backslash u^1_{s+1}$. Since M^2_i is a limit model for each $i, i+1 \in u^2_{s+1}\backslash u^1_{s+1}$, we can apply Theorem 7.2 to find $N^2_i \preceq_{\mathcal{K}} M^2_i$ such that ga-tp(a_i/M^2_i) does not μ-split over N^2_i and M^2_i is universal over N^2_i.

Define $f_{s,s+1} := g{\circ}h \restriction \bigcup_{i\in u^2_s} M^2_i$ and $\check{f}_{s,s+1} := g{\circ}h$. To complete the definition of a directed system, for every $r \leq s$, set $f_{r,s+1} := f_{s,s+1}{\circ} f_{r,s}$ and $\check{f}_{r,s} := \check{f}_{s,s+1}{\circ}\check{f}_{r,s}$.

t is a limit ordinal: Suppose that $(\langle\bigcup_{i\in u^2_s} M^2_i(= M^s) \mid s < t\rangle, \langle f_{r,s} \mid r \leq s < t\rangle)$ and $(\langle \check{M} \mid s < t\rangle, \langle \check{f}_{r,s} \mid r \leq s < t\rangle)$ have been defined. Since these are both directed systems, we can take direct limits. By niceness, we can apply Claim II.6.13, so that we may assume that $(M^*, \langle f^*_{s,t} \mid s \leq t\rangle)$ and $(\check{M}, \langle \check{f}^*_{s,t} \mid s \leq t\rangle)$ are respective direct limits such that $M^* \prec_{\mathcal{K}} \check{M}$, $\check{f}^*_{s,t} \supset f^*_{s,t}$ and $\bigcup_{s<t}\bigcup_{i\in u^1_s} M^1_i \prec_{\mathcal{K}} M^*$.

By condition (4) of the construction, notice that M^* is a (μ, t)-limit model witnessed by $\langle f^*_{s,t}(M^s) \mid s < t\rangle$. Hence M^*_t is an amalgamation base. As in the successor case of the construction in the proof of Theorem II.7.11, we can find $\bar{M}' = \langle M'_i \mid i \in \bigcup_{s<t} \mathfrak{U}^2_s \bigcup u^1_t\rangle$ and an automorphism h of $\check{M}$ such that

- $(\bar{M}', \bar{a}', \bar{N}')$ is a nice scattered tower, where $\bar{a}' = \langle a^2_i \mid i \in \bigcup_{s<t} \mathfrak{U}^2_s \bigcup u^1_t\rangle$ and $\bar{N}' = \langle N^2_i \mid i \in \bigcup_{s<t} \mathfrak{U}^2_s \bigcup u^1_t\rangle$
- $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \restriction \mathfrak{U}^1_t <^c (\bar{M}', \bar{a}', \bar{N}')$

- $\bigcup_{i\in\bigcup_{s<t}\mathfrak{U}^2_s\bigcup u^1_t} M'_i \cap \{a^1_j \mid j \in \mathfrak{U}^1 \setminus \mathfrak{U}^1_t\} = \emptyset$.
- $h \restriction M^* : M^* \cong M'_{\min\{u^1_t\}}$ and
- $h \restriction M^1_{\min\{u^1_t\}} = id_{M^1_{\min\{u^1_t\}}}$.

Let $M^\dagger$ be a $(\mu, |u^2_t \backslash u^1_t| + \aleph_0)$-limit model over $\bigcup_{i\in\bigcup_{s<t}\mathfrak{U}^2_s\bigcup u^1_t} M'_i$ such that such that $M^1_{\min\{u^2_{t+1}\}} \prec_{\mathcal{K}} M^\dagger \prec_{\mathcal{K}} \check{M}$. Let $\langle M^\dagger_\gamma \mid \gamma < |u^2_{s+1}\backslash u^1_{s+1}| + \aleph_0\rangle$ witness that $M^\dagger$ is a limit model. Since limit models are amalgamation bases, we may choose $M^\dagger_{\gamma+1}$ to be a (μ,ω)-limit over $M^\dagger_\gamma$.

Applying Corollary II.5.3 to $(\bigcup_{i\in u^1_t} M^1_i, \bigcup_{i\in\bigcup_{s<t}\mathfrak{U}^2_s\bigcup u^1_t} M'_i, M^\dagger)$ and $\{a^1_j \mid j \in \bigcup \mathfrak{U}^1 \setminus \mathfrak{U}^1_t\}$, there exists an automorphism of $\check{M}$, g, such that

- $g \restriction \bigcup_{i\in u^1_t} M^1_i = id_{\bigcup_{i\in u^1_t} M^1_i}$ and
- $g(M^\dagger) \cap \{a^1_j \mid j \in \bigcup \mathfrak{U}^1 \setminus \mathfrak{U}^1_t\} = \emptyset$.

Denote by $\langle i_\gamma \mid \gamma \in \mathrm{otp}(u^2_t\backslash u^1_t)\rangle$ the increasing enumeration of $u^2_t\backslash u^1_t$. Define

$$M^2_i := \begin{cases} g(M'_i) & \text{for } i \in u^1_t \\ g(M^\dagger_\gamma) & \text{for } i = i_\gamma \in u^2_t\backslash u^1_t \end{cases}$$

Since $M^\dagger$ is a limit model witnessed by the $M^\dagger_\gamma$'s, we can choose $a_i \in M^2_{i+1}\backslash M^2_i$ for all $i, i+1 \in u^2_t\backslash u^1_t$. Since M^2_i is a limit model for each $i, i+1 \in u^2_t\backslash u^1_t$, we can apply Theorem 7.2 to find $N^2_i \preceq_{\mathcal{K}} M^2_i$ such that ga-tp(a_i/M^2_i) does not μ-split over N^2_i and M^2_i is universal over N^2_i.

Define $f_{s,t} := g \circ h \circ f^*_{s,t} \restriction \bigcup_{i\in u^2_s} M^2_i$ and $\check{f}_{s,t} := g \circ h \circ f^*_{s,t}$ for all $s < t$.

$\dashv$

Notice that in the proof of the $<^c$-extension property for nice towers, we have actually shown that there is some freedom in choosing the new $a'_i s$:

Corollary II.8.9. *Let $\mathfrak{U}^1$ and $\mathfrak{U}^2$ be sets of intervals of ordinals $< \mu^+$ such that $\mathfrak{U}^2$ is an interval extension of $\mathfrak{U}^1$. Let $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}^1}$ be a nice scattered tower. There exists a nice scattered tower $(\bar{M}^2, \bar{a}^2, \bar{N}^2) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}^2}$ such that $(\bar{M}^1, \bar{a}^1, \bar{N}^1) <^c (\bar{M}^2, \bar{a}^2, \bar{N}^2)$. Moreover for every $i \in \bigcup \mathfrak{U}^2 \setminus \bigcup \mathfrak{U}^1$ and every $j < i$ with $j \in \mathfrak{U}^1$, if $(p, N) \in \mathfrak{St}(M^1_j)$, then we can choose a_i such that $(p, N) \sim (\text{ga-tp}(a^2_i/M^2_i), N^2_i) \restriction M^1_j$.*

If we isolate the induction step, we get the following useful fact:

Corollary II.8.10. *Suppose $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\mathfrak{U}}$ lies inside a (μ, μ^+)-limit model, $\check{M}$, that is $\bigcup_{i<\alpha} M_i \prec_{\mathcal{K}} \check{M}$. If for some $\mathfrak{U}' \subset_{int} \mathfrak{U}$, $(\bar{M}', \bar{a}', \bar{N}') \in \mathcal{K}^*_{\mu,\mathfrak{U}'}$ is a partial extension of $(\bar{M}, \bar{a}, \bar{N})$ (ie $(\bar{M}, \bar{a}, \bar{N}) \restriction \mathfrak{U} \cap \beta <^c (\bar{M}', \bar{a}', \bar{N}')$), then there exist a $\mathcal{K}$-mapping f, models $M'_{\sup\{\bigcup \mathfrak{U}'\}+1}$ and $N'_{\sup\{\bigcup \mathfrak{U}'\}+1}$ and an element $a'_{\sup\{\bigcup \mathfrak{U}'\}}$ such that $f : \bigcup_{i\in\mathfrak{U}'} M'_i \to \check{M}$, $f \restriction M_j = id_{M_j}$ for $j \in \mathfrak{U}'$ and there exists $M'_{\sup\{\bigcup \mathfrak{U}'\}+1} \in \mathcal{K}^*_\mu$ so that $(\langle f(M'_i) \mid i \in \bigcup \mathfrak{U}' \rangle \hat{} \langle M'_{\sup\{\bigcup \mathfrak{U}'\}+1} \rangle, \langle a'_i \mid i \in \bigcup \mathfrak{U}' \rangle \hat{} \langle a'_{\sup\{\bigcup \mathfrak{U}'\}+1} \rangle, \langle f(N'_i) \mid i \in \bigcup \mathfrak{U}' \rangle \hat{} \langle N'_{\sup\{\bigcup \mathfrak{U}'\}+1} \rangle,)$ is a partial $<^c_{\mu,j+2}$ extension of $(\bar{M}, \bar{a}, \bar{N})$.*

2.9 Reduced Towers are Continuous

In Section 2.10 we identify a property (full and continuous) which will guarantee that for a tower $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ with this property, we have that $\bigcup_{i<\alpha} M_i$ is a (μ, α)-limit model over M_0 (see Theorem II.10.5). This addresses problem (1) in our construction of an array of models described at the beginning of Section 2.8. The first point that (1) breaks down is that $\langle M^{\theta_2}_i \mid i < \theta_1 \rangle$ need not be a continuous chain of models, since we do not require towers to be continuous. Shelah and Villaveces introduced the concept of reduced towers in an attempt to capture some continuous towers. Unfortunately, their proof that reduced towers are continuous does not converge. Here we solve this problem. We introduce a strengthening of reduced towers, completely reduced towers, for easier reading.

Definition II.9.1. A tower $(\bar{M}, \bar{a}, \bar{N}) \in^{+} \mathcal{K}^{*}_{\mu,\mathfrak{U}}$ is said to be *reduced* provided that for every $(\bar{M}', \bar{a}', \bar{N}') \in {}^{+}\mathcal{K}^{*}_{\mu,\mathfrak{U}}$ with $(\bar{M}, \bar{a}, \bar{N}) \leq^{c} (\bar{M}', \bar{a}', \bar{N}')$ we have that for every $i \in \bigcup \mathfrak{U}$,

$$M'_i \cap \bigcup_{j \in \bigcup \mathfrak{U}} M_j = M_i.$$

Definition II.9.2. A tower $(\bar{M}, \bar{a}, \bar{N}) \in^{+} \mathcal{K}^{*}_{\mu,\mathfrak{U}}$ is said to be *completely reduced* provided that for every $\zeta \leq \sup\{\bigcup \mathfrak{U}\}$ and every $(\bar{M}', \bar{a}', \bar{N}') \in {}^{+}\mathcal{K}^{*}_{\mu,\mathfrak{U}\cap\zeta}$ with $(\bar{M}, \bar{a}, \bar{N}) \restriction \mathfrak{U}\cap\zeta \leq^{c} (\bar{M}', \bar{a}', \bar{N}')$ we have that for every $i \in \bigcup \mathfrak{U}\cap\zeta$,

$$M'_i \cap \bigcup_{j \in \bigcup \mathfrak{U}\cap\zeta} M_j = M_i.$$

Proposition II.9.3. *If $(\bar{M}, \bar{a}, \bar{N})$ is reduced, then it is completely reduced.*

Proof. Suppose that $(\bar{M}, \bar{a}, \bar{N})$ is not completely reduced, then there exist a $\zeta < \sup\{\mathfrak{U}\}$, a tower $(\bar{M}', \bar{a}', \bar{N}') \in {}^{+}\mathcal{K}^{*}_{\mu,\mathfrak{U}\restriction\zeta}$, $i \in \bigcup \mathfrak{U}\cap\zeta$ and an element b such that

- $(\bar{M}, \bar{a}, \bar{N}) \restriction (\mathfrak{U} \restriction \zeta) \leq^{c} (\bar{M}', \bar{a}', \bar{N}')$ and
- $b \in (M'_i \cap \{\bigcup_{j \in \bigcup \mathfrak{U}\cap\zeta} M_j) \backslash M_i$.

By Lemma II.8.10, there exists a $\mathcal{K}$-mapping f and a tower $(\bar{M}^{*}, \bar{a}^{*}, \bar{N}^{*}) \in {}^{+}\mathcal{K}^{*}_{\mu,\mathfrak{U}}$ such that

(1) $(\bar{M}, \bar{a}, \bar{N}) \leq^{c} (\bar{M}^{*}, \bar{a}^{*}, \bar{N}^{*})$,

(2) $f : \bigcup_{j \in \bigcup \mathfrak{U}\cap\zeta} M'_i \to \bigcup_{j \in \bigcup \mathfrak{U}\cap\zeta} M^{*}_j$,

(3) $f \restriction \bigcup_{j \in \bigcup \mathfrak{U}\cap\zeta} M_i = id_{\bigcup_{j \in \bigcup \mathfrak{U}\cap\zeta} M_i}$,

(4) for every $j \in \bigcup \mathfrak{U} \cap \zeta$, $f(M'_j) = M^*_j$

Notice that by (3) and the fact that $b \in \bigcup_{j \in \bigcup \mathfrak{U} \cap \zeta} M_j$, we have that $f(b) = b$. Since $b \in M'_i$, we have $b \in f(M'_i) = M^*_i$. Thus $(\bar{M}^*, \bar{a}^*, \bar{N}^*)$ witnesses that $(\bar{M}, \bar{a}, \bar{N})$ is not reduced.

⊣

Corollary II.9.4. *If $(\bar{M}, \bar{a}, \bar{N}) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}}$ is reduced, then for every $\zeta < \sup\{\bigcup \mathfrak{U}\}$, $(\bar{M}, \bar{a}, \bar{N}) \restriction \zeta$ is also reduced.*

Proof. Immediate from the definitions and Proposition II.9.3. ⊣

If we take a $<^c$-increasing and continuous chain of reduced towers with increasing index sets, the union will be reduced. The following proposition appears in [ShVi] for the special case when $\mathfrak{U} = \{\alpha\}$ for some limit ordinal α (Theorem 3.1.14 of [ShVi]. We provide the proof here for completeness.

Proposition II.9.5. *Let $\langle \mathfrak{U}_\gamma \mid \gamma < \beta \rangle$ be an increasing and continuous sequence of sets of intervals ($\mathfrak{U}_{\gamma+1}$ is an interval-extension of $\mathfrak{U}_\gamma$ and if γ is a limit ordinal $\bigcup \mathfrak{U}_\gamma = \bigcup_{\delta<\gamma} \bigcup \mathfrak{U}_\delta$.) If $\langle (\bar{M}, \bar{a}, \bar{N})^\gamma \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}_\gamma} \mid \gamma < \beta \rangle$ is $<^c$-increasing and continuous sequence of reduced towers, then the union of these towers is reduced.*

Proof. Denote by $(\bar{M}, \bar{a}, \bar{N})^\beta$ the limit of the sequence of towers and $\mathfrak{U}_\beta$ the limit of the intervals. More specifically, $\mathfrak{U}_\beta$ is a fixed set of intervals such that $\bigcup \mathfrak{U}_\beta = \bigcup_{\gamma<\beta} \bigcup \mathfrak{U}_\gamma$ and for every $\gamma < \beta$, $\mathfrak{U}_\beta$ is an interval extension of $\mathfrak{U}_\gamma$. $\bar{M}^\beta = \langle M^\beta_i \mid i \in \bigcup \mathfrak{U}_\beta \rangle$ where $M^\beta_i = \bigcup_{\{\gamma<\beta \mid i \in \bigcup \mathfrak{U}_\gamma\}} M^\gamma_i$. $\bar{N}^\beta = \langle N_i^{\min\{\gamma \mid i \in \bigcup \mathfrak{U}_\gamma\}} \mid i \in \bigcup \mathfrak{U}_\beta \rangle$ and $\bar{a}^\beta = \langle a_i^{\min\{\gamma \mid i \in \bigcup \mathfrak{U}_\gamma\}} \mid i \in \bigcup \mathfrak{U}_\beta \rangle$

Suppose that it is not reduced. Let $(\bar{M}', \bar{a}, \bar{N}) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}_\beta}$ witness this. Then there exists an $i \in \bigcup \mathfrak{U}_\beta$ and an element a such that $a \in (M'_i \cap \bigcup_{j \in \mathfrak{U}_\beta} M^\beta_j) \backslash M^\beta_i$.

There exists $\gamma < \beta$ such that $i \in \mathfrak{U}_\gamma$ and there exists $j \in \mathfrak{U}_\gamma$ such that $a \in M^\gamma_j$. Now consider the tower in ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}_\gamma}$, $(\bar{M}', \bar{a}, \bar{N}) \restriction \mathfrak{U}_\gamma$. Notice that $(\bar{M}', \bar{a}, \bar{N}) \restriction \mathfrak{U}_\gamma$ witnesses that $(\bar{M}, \bar{a}, \bar{N})^\gamma$ is not reduced. $\dashv$

The following proposition will be used in conjuntion with Theorem II.9.7 to show that every tower can be properly extended to a continuous tower. It appears in [ShVi] (Theorem 3.1.13) for the particular case of $\mathfrak{U} = \{\alpha\}$ for limit ordinals α. John Baldwin has asked for us to elaborate on their proof here. We provide a proof of the more general result with $\mathfrak{U}$ an arbitrary set of intervals on $\alpha < \mu^+$.

Proposition II.9.6 (Density of reduced towers). *Let* $(\bar{M}, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ *be nice. Fix* $\check{M}$ *a* (μ, μ^+)*-limit model containing* $\bigcup_{i\in\mathfrak{U}} M_i$. *Then there exists* $(\bar{M}', \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ *such that*

- $(\bar{M}, \bar{a}, \bar{N}) <^c (\bar{M}', \bar{a}, \bar{N})$,
- $(\bar{M}', \bar{a}, \bar{N})$ *is reduced and*
- $\bigcup_{i\in\bigcup\mathfrak{U}} M_i \prec_{\mathcal{K}} \check{M}$.

Proof. We first observe that it suffices to find a $<^c$-extension, $(\bar{M}', \bar{a}', \bar{N}')$, of $(\bar{M}, \bar{a}, \bar{N})$ that is reduced. If $(\bar{M}', \bar{a}', \bar{N}')$ does not lie inside of $\check{M}$, since $(\bar{M}, \bar{a}, \bar{N})$ is nice, we can apply Proposition II.2.33 to find a $\prec_{\mathcal{K}}$-mapping $f : \bigcup_{i\in\bigcup\mathfrak{U}} M'_i \to \check{M}$ such that $f \restriction \bigcup_{i\in\bigcup\mathfrak{U}} M_i$. Notice that $f[(\bar{M}', \bar{a}', \bar{N}')]$ is as required.

Suppose for the sake of contradiction that no $\leq^c$-extension of $(\bar{M}, \bar{a}, \bar{N})$ in ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}}$ is reduced. This allows us to construct a $\leq^c$-increasing and continuous sequence of towers $\langle(\bar{M}^\zeta, \bar{a}^\zeta, \bar{N}^\zeta) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}} \mid \zeta < \mu^+\rangle$ such that $(\bar{M}^{\zeta+1}, \bar{a}^{\zeta+1}, \bar{N}^{\zeta+1})$ witnesses that $(\bar{M}^\zeta, \bar{a}^\zeta, \bar{N}^\zeta)$ is not reduced for $\zeta > 0$.

The construction: Since $(\bar{M}, \bar{a}, \bar{N})$ is nice, we can apply Theorem II.8.7 to find

$(\bar{M}, \bar{a}, \bar{N})^1$ a $<^c$ extension of $(\bar{M}, \bar{a}, \bar{N})$. By our assumption on $(\bar{M}, \bar{a}, \bar{N})$, we know that $(\bar{M}, \bar{a}, \bar{N})^1$ is not reduced.

Suppose that $(\bar{M}, \bar{a}, \bar{N})^\zeta$ has been defined. Since it is a $\leq^c$-extension of $(\bar{M}, \bar{a}, \bar{N})$, we know it is not reduced. By the definition of reduced towers, there must exist a $(\bar{M}, \bar{a}, \bar{N})^{\zeta+1} \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}}$ a $\leq^c$-extension of $(\bar{M}, \bar{a}, \bar{N})^\zeta$, witnessing that $(\bar{M}, \bar{a}, \bar{N})^\zeta$ is not reduced.

For ζ a limit ordinal, let $(\bar{M}, \bar{a}, \bar{N})^\zeta = \bigcup_{\gamma<\zeta}(\bar{M}, \bar{a}, \bar{N})^\gamma$. This completes the construction.

For each $b \in \bigcup_{\zeta<\mu^+, i\in\bigcup\mathfrak{U}} M_i^\zeta$ define

$$i(b) := \min\big\{i \in \bigcup \mathfrak{U} \mid b \in \bigcup_{\zeta\in\mu^+} \bigcup_{\substack{j<i \\ j\in\bigcup\mathfrak{U}}} M_j^\zeta\big\} \text{ and}$$

$$\zeta(b) := \min\big\{\zeta < \mu^+ \mid b \in M_{i(b)}^\zeta\big\}.$$

$\zeta(\cdot)$ can be veiwed as a function from μ^+ to μ^+. Thus there exists a club $E = \{\delta < \mu^+ \mid \forall b \in \bigcup_{i\in\bigcup\mathfrak{U}} M_i^\delta,\ \zeta(b) < \delta\}$. Actually, all we need is for E to be non-empty.

Fix $\delta \in E$. By construction $(\bar{M}^{\delta+1}, \bar{a}^{\delta+1}, \bar{N}^{\delta+1})$ witnesses the fact that $(\bar{M}^\delta, \bar{a}^\delta, \bar{N}^\delta)$ is not reduced. So we may fix $i \in \bigcup\mathfrak{U}$ and $b \in M_i^{\delta+1} \cap \bigcup_{j\in\bigcup\mathfrak{U}} M_j^\delta$ such that $b \notin M_i^\delta$. Since $b \in M_i^{\delta+1}$, we have that $i(b) \leq i$. Since $\delta \in E$, we know that there exists $\zeta < \delta$ such that $b \in M_{i(b)}^\zeta$. Because $\zeta < \delta$ and $i(b) < i$, this implies that $b \in M_i^\delta$ as well. This contradicts our choice of i and b witnessing the failure of $(\bar{M}^\delta, \bar{a}^\delta, \bar{N}^\delta)$ to be reduced. $\dashv$

The following theorem was claimed in [ShVi]. Unfortunately, their proof does not converge. We resolve their problems here.

Theorem II.9.7 (Reduced towers are continuous). *For every* $\alpha < \mu^+ < \lambda$ *and every sequence of ordinals* $\mathfrak{U}$ *on* α, *if* $(\bar{M}, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ *is reduced, then* $\bar{M}$ *is continuous.*

Proof. Let μ be given. Suppose the claim fails for μ and δ is the minimal limit ordinal for which it fails. More precisely, δ is the minimal element of

$$S = \left\{ \delta < \mu^+ \;\middle|\; \begin{array}{l} \delta \text{ is a limit ordinal} \\ \text{there exist } \mathfrak{U} \text{ a sequence of ordinals} \\ \text{and a reduced tower } (\bar{M}, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}} \text{ such that} \\ \sup\{\cup \mathfrak{U}\} \cap \delta = \delta, \\ \delta \in \bigcup \mathfrak{U} \text{ and} \\ M_\delta \neq \bigcup_{i \in (\cup \mathfrak{U}) \cap \delta} M_i \end{array} \right\}.$$

Let $\mathfrak{U}$ be a set of intervals and $(\bar{M}, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ witness $\delta \in S$. Let $b \in M_\delta \setminus \bigcup_{i \in (\cup \mathfrak{U}) \cap \delta} M_i$ be given. Our goal is to arrive to a contradiction by showing that $(\bar{M}, \bar{a}, \bar{N})$ is not completely reduced. By Corollary II.9.4, it is enough to show that $(\bar{M}, \bar{a}, \bar{N}) \restriction (\delta + 1)$ is not reduced. We will find a $\leq^c$-extension $(\bar{M}^*, \bar{a}^*, \bar{N}^*)$ of $(\bar{M}, \bar{a}, \bar{N}) \restriction (\delta + 1)$ such that $b \in M^*_\zeta$ for some $\zeta < \delta$.

Fix $\check{M}$ a (μ, μ^+)-limit over M_δ. We begin by defining by induction on $\zeta < \delta$ a $<^c$-increasing and continuous sequence of reduced towers, $\langle (\bar{M}, \bar{a}, \bar{N})^\zeta \in {}^+ \mathcal{K}^*_{\mu,\mathfrak{U}\restriction\delta} \mid \zeta < \delta \rangle$, such that $(\bar{M}, \bar{a}, \bar{N})^0 \restriction \delta = (\bar{M}, \bar{a}, \bar{N})$ and $M^\zeta_i \prec_{\mathcal{K}} \check{M}$ for all $\zeta < \delta$ and for all $i \in \bigcup \mathfrak{U} \cap \delta$. Why is this possible? By the minimality of δ and Corollary II.9.4, $(\bar{M}, \bar{a}, \bar{N})^0 \restriction \delta$ is continuous. Therefore, it is nice. This allows us to apply Proposition II.9.6 to get a reduced extension $(\bar{M}, \bar{a}, \bar{N})^1$ inside $\check{M}$. Similarly we can find reduced extensions at successor stages. When ζ is a limit ordinal, we take unions which will be reduced by Proposition II.9.5.

Consider the diagonal sequence $\langle M^\zeta_\zeta \mid \zeta \in \bigcup\mathfrak{U}$ and $\zeta < \delta\rangle$. Notice that this is a $\prec_{\mathcal{K}}$-increasing sequence of amalgamation bases and $M^{\zeta'}_{\zeta'}$ is universal over M^ζ_ζ whenever $\zeta < \zeta' \in \bigcup\mathfrak{U}\cap(\delta)$. By minimality of δ, the sequence $\langle M^\zeta_\zeta \mid \zeta \in \bigcup\mathfrak{U}$ and $\zeta < \delta\rangle$ is continuous:

$$\text{for } \zeta \in \bigcup\mathfrak{U}\cap\delta \text{ with } \zeta = \sup\{\bigcup\mathfrak{U}\cap\zeta\},\ M^\zeta_\zeta = \bigcup_{\xi<\zeta} M^\xi_\xi.$$

Thus $\bigcup_{\zeta\in\bigcup\mathfrak{U}\cap\delta} M^\zeta_\zeta$ is a limit model. Since $\bigcup_{\zeta\in\bigcup\mathfrak{U}\cap\delta} M^\zeta_\zeta$ and M_δ are amalgamation bases inside $\check{M}$, we can fix $M^\delta_\delta \prec_{\mathcal{K}} \check{M}$ a (μ,ω)-limit model universal over both $\bigcup_{\zeta\in\bigcup\mathfrak{U}\cap\delta} M^\zeta_\zeta$ and M_δ. (ω was an arbitrary choice, we only need that M^δ_δ be a (μ,θ)-limit for some limit $\theta < \mu^+$.)

Because $\bigcup_{\zeta\in\bigcup\mathfrak{U}\cap\delta} M^\zeta_\zeta$ is a limit model, we can apply Theorem II.7.2 to ga-tp$(b/\bigcup_{\zeta\in\bigcup\mathfrak{U}\cap\delta} M^\zeta_\zeta, M^\delta_\delta$ Let $\xi \in \bigcup\mathfrak{U}\cap\delta$ be such that

$$(*)_1 \quad \text{ga-tp}(b/\bigcup_{\zeta\in\bigcup\mathfrak{U}\cap\delta} M^\zeta_\zeta, M^\delta_\delta) \text{ does not } \mu\text{-split over } M^\xi_\xi.$$

We chose by induction on $i \leq \delta$ a $\prec_{\mathcal{K}}$-increasing and continuous chain of models $\langle N^*_i \in \mathcal{K}^*_\mu \mid i \in \bigcup\mathfrak{U}\cap(\delta+1)\rangle$ and an increasing and continuous sequence of $\mathcal{K}$-mappings $\langle h_i \mid i \in \bigcup\mathfrak{U}\cap(\delta+1)\rangle$ satisfying

(1) $h_i : M^i_i \to N^*_i$ for $i < \delta$

(2) $h_{i+1}(a_i) \notin N^*_i$ for $i, i+1 \in \bigcup\mathfrak{U}\cap(\delta+1)$

(3) $N^*_i \prec_{\mathcal{K}} \check{M}$

(4) N^*_i is universal over N^*_j for $j < i$

(5) $M^\delta_\delta \subseteq N^*_i$ for $i > \xi$

(6) $h_\xi = id_{M^\xi_\xi}$,

(7) ga-tp$(b/h_i(M^i_i))$ does not μ-split over M^ξ_ξ for $i \in \bigcup\mathfrak{U}\cap\delta$ with $i \geq \xi$ and

(8) ga-tp$(h_{i+1}(a_i)/N^*_i)$ does not μ-split over $h_i(N_i)$ for $i, i+1 \in \bigcup \mathfrak{U} \cap (\delta+1)$.

Fix an increasing enumeration of $\bigcup \mathfrak{U} \cap (\delta+1) = \{i_\zeta \mid \zeta \leq \alpha\}$ for some $\alpha \leq \delta$. We construct this sequence of models and sequence of mappings by induction on $\zeta \leq \alpha$. Let ξ^* be such that $\xi^* = i^*_\zeta$:

$\zeta \leq \xi^*$: Set $N^*_{i_\zeta} := M^{i_\zeta}_{i_\zeta}$ and $h_{i_\zeta} = id_{M^{i_\zeta}_{i_\zeta}}$.

$\zeta > \xi^*$ is a limit ordinal and $i_\zeta = \sup\{i_\gamma \mid \gamma < \zeta\}$: To maintain continuity, $N^*_{i_\zeta} := \bigcup_{\gamma<\zeta} N^*_{i_\gamma}$ and $h_{i_\zeta} := \bigcup_{\gamma<\zeta} h_{i_\gamma}$. Condition (7) follows from the induction hypothesis and Theorem II.7.3.

$\zeta > \xi^*$ is a limit ordinal with $i_\zeta \neq \sup\{i_\gamma \mid \gamma < \zeta\}$ or $\zeta = \gamma + 1$ with $i_\zeta \neq i_\gamma + 1$: Let $N^* := \bigcup_{\beta<\zeta} N^*_{i_\beta}$ and $M^* := \bigcup_{\beta<\zeta} M^{i_\beta}_{i_\beta}$. Let $N^{**}_{i_\zeta} \in \mathcal{K}^*_\mu$ be a universal extension of N^* and M^δ_δ with $N^{**}_{i_\zeta} \prec_{\mathcal{K}} \check{M}$. This is possible because either $N^* = N^*_{i_\beta}$ for some β and is therefore a limit model by the induction hypothesis, or continuity and condition (4) guarantee that N^* is a limit model witnessed by $\langle N^*_{i_\beta} \mid \beta < \zeta\rangle$. $N^{**}_{i_\zeta}$ will be a first approximation for our definition of $N^*_{i_\zeta}$. To get condition (7) notice that by the induction hypothesis we have for every $\beta < \zeta$,

$$\text{ga-tp}(b/h_\beta(M^{i_\beta}_{i_\beta})) \text{ does not } \mu\text{-split over } M^\xi_\xi).$$

With an application of Theorem II.7.3, we can conclude that

$$\text{ga-tp}(b/M^*) \text{ does not } \mu\text{-split over } M^\xi_\xi).$$

By Theorem II.7.5 we can find $f \in \operatorname{Aut}_{\bigcup_{\beta<\zeta} h_{i_\beta}(M^{i_\beta}_{i_\beta})}(\check{M})$ such that

$$\text{ga-tp}(b/f(N^{**}_{i_\zeta})) \text{ does not } \mu\text{-split over } M^\xi_\xi).$$

Let $N^*_{i_\zeta} := f(N^{**}_{i_\zeta})$ and $h_{i_\zeta} := f$. Notice that we do not have to concern ourselves with condition (8) since $i_\zeta \neq i_\gamma + 1$. It is routine to verify that $N^*_{i_\zeta}$ and h_{i_ζ} meet the other conditions.

$\zeta = \gamma + 1 > \zeta^*$ *with* $i_\zeta = i_\gamma + 1$: Let $\check{h}_{i_\gamma} \in \mathrm{Aut}(\check{M})$ extend h_{i_γ}. Let $N^{**} \in \mathcal{K}^*_\mu$ be a universal extension of $N^*_{i_\gamma}$, $\check{h}_{i_\gamma}(M^{i_\zeta}_{i_\zeta})$ and M^δ_δ with $N^{**} \prec_{\mathcal{K}} \check{M}$. This will be our first approximation to $N^*_{i_\zeta}$.

We will first work towards condition (2). By Corollary II.5.3, applied to $h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})$, $h_{i_\gamma}(M^{i_\zeta}_{i_\zeta})$, N^{**} and the collection of elements $(M^\delta_\delta \bigcup N^*_{i_\gamma}) \backslash h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})$, we can find a $\mathcal{K}$-mapping f such that

- $f : \check{h}_{i_\gamma}(M^{i_\zeta}_{i_\zeta}) \to N^{**}$
- $f \restriction h_{i_\gamma}(M^{i_\gamma}_{i_\gamma}) = id_{h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})}$ and
- $f(\check{h}_{i_\gamma}(M^{i_\zeta}_{i_\zeta})) \cap (M^\delta_\delta \bigcup N^*_{i_\gamma}) \backslash h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})$ in particular $f \circ \check{h}_{i_\gamma}(a_j) \notin N^*_{i_\gamma}$ for $j \geq i_\gamma$.

Now that we have met condition (2), we focus on meeting condition (8) without mapping a_{i_γ} into $N^*_{i_\gamma}$. By the definition of towers, we have

$$\text{ga-tp}(a_{i_\gamma}/M^{i_\gamma}_{i_\gamma}) \text{ does not } \mu\text{-split over } N^{i_\gamma}_{i_\gamma}.$$

By invariance we have that

$$\text{ga-tp}(f \circ \check{h}_{i_\gamma}(a_{i_\gamma})/h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})) \text{ does not } \mu\text{-split over } h_{i_\gamma}(N^{i_\gamma}_{i_\gamma}).$$

By the extension property for non-splitting (Theorem II.7.5), we can find $g \in \mathrm{Aut}_{h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})}(\check{M})$ such that

$$(*)_2 \quad \text{ga-tp}(g \circ f \circ \check{h}_{i_\gamma}(a_{i_\gamma})/N^*_{i_\gamma}) \text{ does not } \mu\text{-split over } h_{i_\gamma}(N^{i_\gamma}_{i_\gamma}).$$

Let $g' := g \circ f \circ \check{h}_{i_\gamma}$. We need to verify that by applying g' our work towards condition (2) is not lost:

Claim II.9.8. $g'(a_{i_\gamma}) \notin N^*_{i_\gamma}$.

Proof. Since $h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})$ is universal over $h_{i_\gamma}(N^{i_\gamma}_{i_\gamma})$, there exists a $\mathcal{K}$-mapping $H : N^*_{i_\gamma} \to h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})$ with $H \restriction h_{i_\gamma}(N^{i_\gamma}_{i_\gamma}) = id_{h_{i_\gamma}(N^{i_\gamma}_{i_\gamma})}$. By definition of g' and $(*_2)$, we have ga-tp$(g'(a_{i_\gamma})/N^*_{i_\gamma})$ does not μ-split over $h_{i_\gamma}(N^{i_\gamma}_{i_\gamma})$. Thus

$$(*)_3 \quad \text{ga-tp}(g'(a_{i_\gamma})/H(N^*_{i_\gamma})) = \text{ga-tp}(H(g'(a_{i_\gamma}))/H(N^*_{i_\gamma})).$$

Suppose for the sake of contradiction that $g'(a_{i_\gamma}) \in N^*_{i_\gamma}$. Then an application of H gives us that $H(g'(a_{i_\gamma})) \in H(N^*_{i_\gamma})$. Thus by the above equality of types $(*)_3$), we have that $g'(a_{i_\gamma}) \in H(N^*_{i_\gamma})$. Since rg$(H) \subseteq h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})$ we get that $g'(a_{i_\gamma}) \in h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})$.

Since $a_{i_\gamma} \notin M^{i_\gamma}_{i_\gamma}$ and since $g' \restriction M^{i_\gamma}_{i_\gamma} = h_{i_\gamma}$, an application of g' gives us $g(a_{i_\gamma}) \notin h_{i_\gamma}(M^{i_\gamma}_{i_\gamma})$, contradicting the previous paragraph. $\dashv$

We now tackle condition (7). Fix $N^*_{i_\zeta} \prec_{\mathcal{K}} \check{M}$ such that it is universal over $g'(M^{i_\zeta}_{i_\zeta})$, $N^*_{i_\gamma}$ and N^{**}. By monotonicity of non-splitting $(*)_1$ implies

$$\text{ga-tp}(b/M^{i_\gamma}_{i_\gamma}) \text{ does not } \mu\text{-split over } M^\xi_\xi.$$

By invariance we get

$$\text{ga-tp}(g'(b)/g'(M^{i_\gamma}_{i_\gamma})) \text{ does not } \mu\text{-split over } M^\xi_\xi.$$

By the extension property for non-splitting, we can find $k \in \text{Aut}_{g'(M^{i_\gamma}_{i_\gamma})} \check{M}$ such that

$$\text{ga-tp}(k \circ g'(b)/N^*_{i_\zeta}) \text{ does not } \mu\text{-split over } M^\xi_\xi.$$

Set $h_{i_\zeta} := k \circ g' \restriction N^*_{i_\zeta}$. Since $k \restriction g'(M^{i_\gamma}_{i_\gamma}) = id_{g'(M^{i_\gamma}_{i_\gamma})}$, conditions (2) and (8) are met by h_{i_ζ}. This completes the construction.

The construction is enough: Notice that $i_\alpha = \delta$. Consider the increasing and continuous sequence $\langle h_\delta(M^{i_\gamma}_{i_\gamma}) \mid \gamma < \alpha\rangle$. By invariance, when $i < j$, $h_\delta(M^j_j)$ is universal over $h_\delta(M^i_i)$ and $h_\delta(M^i_i)$ is a limit model. By construction we have that for every $i \in \bigcup \mathfrak{U} \cap \delta$,

$$\text{ga-tp}(b/h_\delta(M^i_i))\text{does not } \mu\text{-split over} M^\xi_\xi.$$

This allows us to apply Theorem II.7.3, to ga-tp$(b/\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} h_\delta(M^i_i))$ to conclude that

$$(*)_4 \quad \text{ga-tp}(b/\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} h_\delta(M^i_i)) \text{ does not } \mu\text{-split over } M^\xi_\xi.$$

Claim II.9.9. *There exists $\check{h} \in \mathrm{Aut}(\check{M})$ extending $\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} h_i$ such that $\check{h}(b) = b$.*

Proof. Notice that $\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} M^i_i$ is a limit model witnessed by $\langle M^j_j \mid j \in \bigcup\mathfrak{U}\cap i\rangle$. So we can apply Proposition II.2.32 and extend $\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} h_i$ to an autmorphism h^* of $\check{M}$. We will first show that

$$(*)_5 \quad \text{ga-tp}(b/h^*(\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} M^i_i), \check{M}) = \text{ga-tp}(h^*(b)/h^*(\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} M^i_i), \check{M}).$$

By invariance and our choice of ξ we have that

$$\text{ga-tp}(h^*(b)/h^*(\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} M^i_i), \check{M}) \text{ does not } \mu\text{-split over } M^\xi_\xi.$$

We will use non-splitting to show that these two types are equal $(*)_5$. In accordance with the definition of splitting, let $N^1 = \bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} M^i_i$, $N^2 = h^*(\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} M^i_i)$ and $p = \text{ga-tp}(b/h^*(\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} M^i_i), \check{M})$. By $(*)_4$, we have that $p \restriction N^2 = h^*(p \restriction N^1)$. In other words, ga-tp$(b/h^*(\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} M^i_i), \check{M}) =$ ga-tp$(h^*(b)/h^*(\bigcup_{i\in\bigcup\mathfrak{U}\cap\delta} M^i_i), \check{M})$, as desired.

From this equality of types $(*)_5$, we can find an automorphism f of $\check{M}$ such that $f(h^*(b)) = b$ and $f \restriction h^*(\bigcup_{i \in \bigcup \mathfrak{U} \cap \delta} M^i_i) = id_{h^*(\bigcup_{i \in \bigcup \mathfrak{U} \cap \delta} M^i_i)}$. Notice that $h := f \circ h^*$ is as required.

$\dashv$

For each $i \leq \delta$ define $M^*_i := h^{-1}(N^*_i)$. Let $\zeta := \min\{i \in \mathfrak{U} \mid i > \xi + 1\}$. Notice that since $\delta = \sup\{\mathfrak{U} \cap \delta\}$ and $\delta > \xi$, we have that $\zeta < \delta$. Let $\mathfrak{U}^* = \mathfrak{U} \cap (\delta + 1)$.

Claim II.9.10. $(\bar{M}^*, \bar{a} \restriction \bigcup \mathfrak{U}^*, \bar{N} \restriction \mathfrak{U}^*)$ *is a* $\leq^c$*-extension of* $(\bar{M}, \bar{a}, \bar{N}) \restriction \bigcup \mathfrak{U}^*$ *such that* $b \in M^*_\zeta$.

Proof. By construction $b \in M^\delta_\delta \subseteq N^*_\zeta$. Since $h(b) = b$, this implies $b \in M^*_\zeta$. To verify that we have a $\leq^c$-extension we need to show for $i \in \mathfrak{U}^*$:

i. $M^*_i = M_i$ or M^*_i is universal over M_i

ii. $a_j \notin M^*_i$ for $j \in \mathfrak{U}^*$ with $j \geq i$ and

iii. ga-tp(a_i/M^*_i) does not μ-split over N_i whenever $i, i+1 \in \bigcup \mathfrak{U}^*$.

Item i. follows from the fact that M^i_i is universal over M_i and $M^i_i \prec_{\mathcal{K}} M^*_i$. Condition (2) of the construction of $\langle N^*_i \mid i \in \bigcup \mathfrak{U} \cap (\delta + 1)\rangle$ guarantees that for $j \geq i$, $h(a_j) \notin N^*_i$. Thus for $j \geq i$, $a_j \notin M^*_i$. iii follows from condition (8) of the construction and invariance. $\dashv$

Notice that $(\bar{M}^*, \bar{a} \restriction \bigcup \mathfrak{U}^*, \bar{N} \restriction \bigcup \mathfrak{U}^*)$ witnesses that $(\bar{M}, \bar{a}, \bar{N}) \restriction \bigcup \mathfrak{U}^*$ is not reduced. This gives us a contradiction and completes the proof of the theorem.

$\dashv$

2.10 Full towers

Definition II.10.1 (Definition 3.2.1 of [ShVi]). For M a (μ, θ)-limit model,

(1) Let

$$\mathfrak{St}(M) := \left\{ (p, N) \;\middle|\; \begin{array}{l} N \prec_{\mathcal{K}} M; \\ N \text{ is a } (\mu,\theta)-\text{limit model}; \\ M \text{ is universal over } N \text{ and} \\ p \in \text{ga-S}(M) \text{ does not } \mu-\text{split over } N. \end{array} \right\}$$

and

(2) For types $(p_l, N_l) \in \mathfrak{St}(M)$ $(l = 1, 2)$, we say $(p_1, N_1) \sim (p_2, N_2)$ iff for every $M' \in \mathcal{K}^{am}_\mu$ extending M there is a $q \in S(M')$ extending both p_1 and p_2 such that q does not μ-split over N_1 and q does not μ-split over N_2.

Notice that $\sim$ is an equivalence relation on $\mathfrak{St}(M)$.

By Fact II.2.19, we have

Fact II.10.2. *For $M \in \mathcal{K}^{am}_\mu$, $|\,\mathfrak{St}(M)/\sim| \leq \mu$.*

We can then consider towers which are saturated with respect to $\mathfrak{St}(M)$:

Definition II.10.3. A tower $(\bar{M}, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ is said to be *full* iff

(1) μ divides $\text{cf}(\sup\{\bigcup \mathfrak{U}\})$ if μ is regular, otherwise μ^ω divides $\text{cf}(\sup\{\bigcup \mathfrak{U}\})$ and

(2) if $\beta \in \bigcup \mathfrak{U}$ and $(p, N^*) \in \mathfrak{St}(M_\beta)$, then for some $i < \mu$ with $\beta + i \in \bigcup \mathfrak{U}$, we have that $(\text{ga-tp}(a_{\beta+i}, M_{\beta+i}, M_{\beta+i+1}), N_{\beta+i}) \sim (p, N^*)$,

by μ dividing α we mean there exists γ such that $\alpha = \gamma \cdot \mu$ where $\cdot$ is ordinal multiplication.

Remark II.10.4. (1) Definition II.10.3 appears in [ShVi] for the special case when $\mathfrak{U} = \{[0, \alpha)\}$ for α a limit ordinal $< \mu^+$ (see Definition 3.2.3 of their paper).

(2) Condition (1) of Definition II.10.3 is used in the proof of Theorem II.10.5

The following theorem is proved in [ShVi] under the particular instance of $\mathfrak{U} = \{[0, \alpha)\}$ for α a limit ordinal $< \mu^+$ (Theorem 3.2.4 of their work). We require the more general result, but the proof is similar to Shelah and Villaveces' argument.

Theorem II.10.5. *If $(\bar{M}, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ is full and $\bar{M}$ is continuous, then $\bigcup_{i \in \bigcup \mathfrak{U}} M_i$ is a $(\mu, \mathrm{cf}(\sup\{\bigcup \mathfrak{U}\}))$-limit model over M_0.*

In addition, we need the following new theorem which is an analog to the statement that the union of $\kappa(T)$-many saturated models is saturated in first order stable theories. We are not implying that fullness is equivalent to saturation, but that the spirit of the results is similar. The following theorem was not stated in [ShVi] and is new:

Theorem II.10.6 (Union of Full Towers is Full). *Let α be a limit ordinal $< \mu^+$ and let $\mathfrak{U}$ be set of intervals such that $|\mathfrak{U}| < \mu^+$ and if μ is regular μ divides $\mathrm{cf}(|\mathfrak{U}|)$ otherwise μ^ω divides $\mathrm{cf}(|\mathfrak{U}|)$. If $\langle(\bar{M}^\beta, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}} \mid \beta < \alpha\rangle$ is a $<^c$-increasing chain of full towers for $\alpha < \mu^+$, then the union is a full tower.*

Proof. Let $\langle(\bar{M}^\beta, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}} \mid \beta < \alpha\rangle$ be a $<^c$-increasing chain of towers. We need to verify that for $i \in \mathfrak{U}$ and $(p, N) \in \mathfrak{S}\mathrm{t}(\bigcup_{\beta<\alpha} M^\beta_i)$, that there exists $j < \mu$ such that $i + j \in \mathfrak{U}$ and $(p, N) \sim (\text{ga-tp}(a_{i+j}, \bigcup_{\beta<\alpha} M^\beta_{i+j}), N_{i+j})$.

By the definition of $<^c$, we have that $\bigcup_{\beta<\alpha} M^\beta_i$ is a (μ, α)-limit witnessed by $\langle M^\beta_i \mid \beta < \alpha\rangle$. By Theorem II.7.2, there exists $\beta < \alpha$ such that p does not

μ-split over M_i^β. Thus $(p \restriction M_i^{\beta+1}, M_i^\beta) \in \mathfrak{St}(M_i^{\beta+1})$. By the assumption of fullness of the $\beta+1^{st}$ tower, there exists a $j < \mu$ such that

$$(p \restriction M_i^{\beta+1}, M_i^\beta) \sim (\text{ga-tp}(a_{i+j}/M_{i+j}^{\beta+1}), N_{i+j}).$$

Recalling the definition of $\sim$, we know that there exists $q \in \text{ga-S}(\bigcup_{\gamma<\alpha} M_{i+j}^\gamma)$ such that

- $p \restriction M_i^\beta \subseteq q$
- $\text{ga-tp}(a_{i+j}/M_{i+j}^{\beta+1}) \subseteq q$
- q does not μ-split over M_i^β and
- q does not μ-split over N_{i+j}.

Notice that it suffices to show

Subclaim II.10.7. $(p, N) \sim (\text{ga-tp}(a_{i+j}/\bigcup_{\gamma<\alpha} M_{i+j}^\gamma), N_{i+j})$.

Proof of Subclaim II.10.7. By definition of $\sim$, we have that

$$(p \restriction M_i^{\beta+1}, M_i^\beta) \sim (p, N).$$

Recalling that $\text{ga-tp}(a_{i+j}/\bigcup_{\gamma<\alpha} M_{i+j}^\gamma)$ does not μ-split over N_{i+j}, we see that

$$(\text{ga-tp}(a_{i+j}/M_{i+j}^{\beta+1}), N_{i+j}) \sim (\text{ga-tp}(a_{i+j}/\bigcup_{\gamma<\alpha} M_{i+j}^\gamma), N_{i+j}).$$

Since $\sim$ is transitive, we have that $(p, N) \sim (\text{ga-tp}(a_{i+j}/\bigcup_{\gamma<\alpha} M_{i+j}^\gamma), N_{i+j})$. $\dashv$

$\dashv$

2.11 Uniqueness of Limit Models

Recall the running assumptions:

(1) $\mathcal{K}$ is an abstract elementary class,

(2) $\mathcal{K}$ has no maximal models,

(3) $\mathcal{K}$ is categorical in some $\lambda > LS(\mathcal{K})$,

(4) GCH and $\Phi_{\mu^+}(S^{\mu^+}_\theta)$ holds for every cardinal $\mu < \lambda$ and every regular θ with $\theta < \mu^+$.

Under these assumptions, we can prove the uniqueness of limit models using the results from Sections 2.8 and 2.9. This is a solution to a conjecture from [ShVi].

Theorem II.11.1 (Uniqueness of Limit Models). *Let μ be a cardinal θ_1, θ_2 limit ordinals such that $\theta_1, \theta_2 < \mu^+ \leq \lambda$. If M_1 and M_2 are (μ, θ_1) and (μ, θ_2) limit models over M, respectively, then there exists an isomorphism $f : M_1 \cong M_2$ such that $f \upharpoonright M = id_M$.*

Proof. Let $M \in \mathcal{K}^{am}_\mu$ be given. By Proposition II.2.29 wlog we may assume that θ_1 and θ_2 are regular. By Proposition II.2.27 it suffices to construct a model which is simultaneously a (μ, θ_1)-limit model and a (μ, θ_2)-limit model over M. Also by Proposition II.2.27 we may assume that if μ is regular μ divides θ_2 otherwise μ^ω divides θ_2. The idea is to build a (scattered) array of models such that at some point in the array, we will find a model which is a (μ, θ_1)-limit model witnessed by its height in the array and is a (μ, θ_2)-limit model witnessed by its horizontal position in the array, fullness and continuity. To guarantee that we have continuous towers, we will be constructing the array with reduced towers. We will define a chain of scattered towers of length $\mu^+ \times \theta_1$ while increasing the index set of the towers as we proceed.

We will consider the index set $\mathfrak{U}^{(\alpha,\zeta)}$ at stage $(\alpha, \zeta) \in \mu^+ \times \theta_1$ where

$$\mathfrak{U}^{(\alpha,\zeta)} := \bigcup \{[\beta\mu, \beta\mu\theta_1 + \mu\zeta) \mid \beta < \alpha\}.$$

Define by induction on $(\alpha, \zeta) \in \mu^+ \times \theta_1$ the $<^c$-increasing and continuous sequence of scattered towers, $\langle (\bar{M}, \bar{a}, \bar{N})^{(\alpha,\zeta)} \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}^{(\alpha,\zeta)}} \mid (\alpha, \zeta) \in \mu^+ \times \theta_1 \rangle$, such that

(1) $M \prec_{\mathcal{K}} M_0^{(\alpha,\zeta)}$,

(2) $(\bar{M}, \bar{a}, \bar{N})^{(\alpha,\zeta)}$ is reduced,

(3) $(\bar{M}, \bar{a}, \bar{N})^{(\alpha,0)} := \bigcup_{\beta<\alpha} \bigcup_{\zeta<\theta_1} (\bar{M}, \bar{a}, \bar{N})^{(\beta,\zeta)}$ and

(4) in successor stages in new intervals of length μ put in representatives of all $\mathfrak{St}$-types from the previous stages, more specifically, if there exists an interval u of length μ in $\mathfrak{U}^{\alpha,\zeta+1} \setminus \mathfrak{U}^{\alpha,\zeta}$, then for every $i \in \bigcup \mathfrak{U}^{\alpha,\zeta}$ with $i < \min\{u\}$ and every $(p, N) \in \mathfrak{St}(M_i)$ there exists $j \in u$ such that $(p, N) \sim (\text{ga-tp}(a_j/M_j), N_j)$.

This construction is possible:

$(0, 0)$: We can choose $\langle M_i^* \mid i \in \mathfrak{U}^{(0,0)} \rangle$ to be an arbitrary $\prec_{\mathcal{K}}$ increasing sequence of limit models of cardinality μ with $M_0^* = M$. For each $i \in \mathfrak{U}^{(0,0)}$ whenever $i+1 \in \mathfrak{U}^{(0,0)}$, fix $a_i^{(0,0)} \in M_{i+1}^* \setminus M_i^*$. Now consider ga-tp$(a_i^{(0,0)}/M_i^*)$. Since $M_i^{(0,0)}$ is a limit model, we can apply Theorem II.7.2 to fix $N_i^{(0,0)} \in \mathcal{K}_\mu^{am}$ such that ga-tp$(a_i^{(0,0)}/M_i^*)$ does not μ-split over $N_i^{(0,0)}$ and M_i^* is universal over $N_i^{(0,0)}$. By Theorem II.9.6, there exists $\bar{M}^{(0,0)}$ such that $(\bar{M}^{(0,0)}, \bar{a}^{(0,0)}, \bar{N}^{(0,0)})$ is a member of ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}^{(0,0)}}$, is a $<^c$-extension of $(\bar{M}^*, \bar{a}^{(0,0)}, \bar{N}^{(0,0)})$ and is reduced.

$(\alpha, 0)$: Take $(\bar{M}, \bar{a}, \bar{N})^{(\alpha,0)} := \bigcup_{\beta<\alpha} \bigcup_{\zeta<\theta_1} (\bar{M}, \bar{a}, \bar{N})^{(\beta,\zeta)}$

$(\alpha, \zeta+1)$: Suppose that $(\bar{M}, \bar{a}, \bar{N})^{(\alpha,\zeta)}$ has been defined. If $\mathfrak{U}^{(\alpha,\zeta)}$ contains no new intervals of length μ, then by Theorem II.9.6, we may take $(\bar{M}, \bar{a}, \bar{N})^{(\alpha,\zeta+1)}$ to be a reduced, $<^c$-extension of $(\bar{M}, \bar{a}, \bar{N})^{(\alpha,\zeta)}$ in ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}^{(\alpha,\zeta)}}$.

Suppose that $\mathfrak{U}^{(\alpha,\zeta+1)}$ contains new intervals $\{u^\beta \mid \beta < \beta' < \mu\}$ each of length μ. Let $u^\beta = \{u^\beta_l \mid l < \mu\}$ be an enumeration of u^β. By Theorem II.9.6 we can find a reduced extension $(\bar{M}, \bar{a}, \bar{N})' \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}^{(\alpha,\zeta+1)}\setminus\{u^\beta|\beta<\beta'\}}$ of $(\bar{M}, \bar{a}, \bar{N})^{(\alpha,\zeta)}$.

By Fact II.10.2, we can enumerate $\bigcup_{j<\min\{u^\beta\}, j\in\mathfrak{U}^{(\alpha,\zeta)}} \mathfrak{St}(M_j)$ as $\{(p,N)^{u^\beta}_l \mid l < \mu\}$. By Corollary II.8.9 and Theorem II.9.6 we can find a reduced extension $(\bar{M}, \bar{a}, \bar{N})^{(\alpha,\zeta+1)} \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}^{(\alpha,\zeta+1)}}$ of $(\bar{M}, \bar{a}, \bar{N})'$ such that for every $l < \mu$ and $\beta < \beta'$, $(p,N)^{u^\beta}_l \sim \text{ga-tp}(a_{u^\beta_l}/M_{u^\beta_l}, N_{u^\beta_l})$. This completes the construction.

Consider the mapping $f : \mu^+ \to \mu^+$ defined by

$$f(\alpha) := \min \left\{ \alpha' \;\middle|\; \begin{array}{l} \text{for every } \beta < \alpha,\ \zeta < \theta_1,\ i \in \bigcup \mathfrak{U}^{(\beta,\zeta)} \text{ and} \\ \text{for every } (p,N) \in \mathfrak{St}(M_i^{(\beta,\zeta)}) \text{ there} \\ \text{exists } \beta' \leq \alpha' \text{ and } j < \mu \text{ such that} \\ (\text{ga-tp}(a_{i+j}/M^{(\beta',0)}_{i+j}), N_{i+j}) \sim (p,N) \end{array} \right\}$$

By condition (4) of the construction, f can be defined. Then there exists a club C such that

$$\delta \in C \Rightarrow f \restriction \delta : \delta \to \delta.$$

Notice that by the definition of f, this implies

$$(*) \quad \delta \in C \cap S^{\mu^+}_{\theta_2} \Rightarrow (\bar{M}, \bar{a}, \bar{N})^{(\delta,0)} \text{ is full.}$$

Pick $\alpha \in C \cap S^{\mu^+}_{\theta_2}$.

Subclaim II.11.2. *We can find $\langle \alpha_\zeta < \mu^+ \mid \zeta \leq \theta_1 \rangle$, an increasing and continuous sequence of ordinals $\geq \alpha$, such that $(\bar{M}, \bar{a}, \bar{N})^{(\alpha_\zeta,0)} \restriction \mathfrak{U}^{(\alpha,0)}$ is full.*

Proof of Subclaim II.11.2. Take $\langle \alpha_\zeta < \mu^+ \mid \zeta \leq \theta_1 \rangle$ to be an increasing and continuous sequence of ordinals $> \alpha$ from C. By definition of f, $(\bar{M}, \bar{a}, \bar{N})^{(\alpha_\zeta, 0)}$ satisfies the second condition of the definition of full towers: if $i \in \bigcup \mathfrak{U}^{(\alpha_\zeta,0)}$ and $(p, N^*) \in \mathfrak{S}\mathrm{t}(M_i^{(\alpha_\zeta,0)})$, then for some $j < \mu$ with $i + j \in \bigcup \mathfrak{U}^{(\alpha_\zeta,0)}$, we have that $(\text{ga-tp}(a_{i+j}^{(\alpha_\zeta,0)}, M_{i+i}^{(\alpha_\zeta,0)}, M_{i+j+1}^{(\alpha_\zeta,0)}), N_{i+j}^{(\alpha_\zeta,0)}) \sim (p, N^*)$.. Since μ divides θ_2 (or μ^ω divides θ_2 when μ is singular) and $\alpha \in S^{\mu^+}_{\theta_2}$, the restriction $(\bar{M}, \bar{a}, \bar{N})^{(\alpha_\zeta,0)} \restriction \mathfrak{U}^{(\alpha,0)}$ satisfies both conditions of the definiton of fullness.

$\dashv$

Fix a sequence as in Subclaim II.11.2. We see that

$$M^* := \bigcup_{\zeta < \theta_1} \bigcup_{i \in \bigcup \mathfrak{U}^{(\alpha,0)}} M_i^{(\alpha_\zeta,0)} = \bigcup_{i \in \bigcup \mathfrak{U}^{(\alpha,0)}} M_i^{(\alpha_{\theta_1},0)}$$

is a (μ, θ_1)-limit over M witnessed by $\langle \bigcup_{i \in \bigcup \mathfrak{U}^{(\alpha,0)}} M_i^{(\alpha_\zeta,0)} \mid \zeta < \theta_1 \rangle$.

Notice that by Subclaim II.11.2, $(\bar{M}, \bar{a}, \bar{N})^{(\alpha_{\theta_1},0)} \restriction \mathfrak{U}^{(\alpha,0)}$ is full. Furthermore, we see that $(\bar{M}, \bar{a}, \bar{N})^{(\alpha_{\theta_1},0)} \restriction \mathfrak{U}^{(\alpha,0)}$ is continuous since $(\bar{M}, \bar{a}, \bar{N})^{(\alpha_{\theta_1},0)}$ is reduced. Now we can apply Theorem II.10.5 to conclude that M^* is a $(\mu, \mathrm{cf}(\sup\{\bigcup \mathfrak{U}^{(\alpha,0)}\}))$-limit model. But by our choice of α, we have that $\mathrm{cf}(\sup\{\bigcup \mathfrak{U}^{(\alpha,0)}\}) = \theta_2$. Thus M^* is also a (μ, θ_2)-limit model over M.

$\dashv$

The above proof implicitly shows the existence of full towers:

Corollary II.11.3. *There exists an interval* $\mathfrak{U}$ *and a tower* $(\bar{M}, \bar{a}, \bar{N}) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}}$ *such that* $(\bar{M}, \bar{a}, \bar{N})$ *is full.*

CHAPTER III

Stable and Tame Abstract Elementary Classes

In this chapter, we explore stability results in the new context of *tame* abstract elementary classes with the amalgamation property. The main result is:

Theorem III.0.4. *Let $\mathcal{K}$ be a tame abstract elementary class satisfying the amalgamation property without maximal models. There exists a cardinal $\mu_0(\mathcal{K})$ such that for every $\mu \geq \mu_0(\mathcal{K})$ and every $M \in \mathcal{K}_{>\mu}$, $A, I \subset M$ such that $|I| \geq \mu^+ > |A|$, if $\mathcal{K}$ is Galois-stable in μ, then there exists $J \subset I$ of cardinality μ^+, Galois-indiscernible sequence over A. Moreover J can be chosen to be a Morley sequence over A.*

This result strengthens Claim 4.16 of [Sh 394] as we do not assume categoricity. This is also an improvement of a result from [GrLe1] concerning the existence of indiscernible sequences.

A step toward this result involves proving:

Theorem III.0.5. *Suppose $\mathcal{K}$ is a tame AEC. If $\mu \geq Hanf(\mathcal{K})$ and $\mathcal{K}$ is Galois μ-stable then $\kappa_\mu(\mathcal{K}) < Hanf(\mathcal{K})$*

This is generalizes a result from [Sh3].

3.1 Introduction

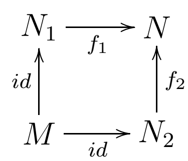
Already in the fifties model theorists studied non-elementary classes of structures (e.g. Jónsson [Jo1], [Jo2] and Fraïssé [Fr]). In [Sh 88], Shelah introduced the framework of abstract elementary classes and embarked on the ambitious program of developing a *classification theory for Abstract Elementary Classes.* While much is known about abstract elementary classes, especially when $\mathcal{K}$ is an AEC under the additional assumption that there exists a cardinal $\lambda > Hanf(\mathcal{K})$ such that $\mathcal{K}$ is categorical in λ, little progress has been made towards a full-fledged stability theory. One of the open problems from [Sh 394] (Remark 4.10(1)) is to identify of a good (forking-like) notion of independence for abstract elementary classes. This is open even for classes that have the amalgamation property and are categorical above the Hanf number. In [Sh 394], several weak notions of independence are introduced under the assumption that the class is categorical. Among these notions is the Galois-theoretic notion of non-splitting. This notion is further developed for categorical abstract elementary classes in Chapter II with the extension property and in [ShVi] with a powerful substitute for $\kappa(T)$ (listed here as Theorem II.7.2). Here we study the notion of non-splitting in a more general context than categorical AEC: *Tame stable classes.* We plan to use Morley sequences for non-splitting as a bootstrap to define a dividing-like concept for these classes.

3.2 Background

Much of the necessary background for this chapter has already been introduced in the Background section of Chapter II. We begin by reviewing the definition of

Galois-type, since we will be considering variations of the underlying equivalence relation E in this chapter.

Definition III.2.1. Let $\beta > 0$ be an ordinal. For triples $(\bar{a}_l, M_l, N_l)$ where $\bar{a}_l \in {}^{\beta}\mathrm{N}_l$ and $M_l \prec_{\mathcal{K}} N_l \in \mathcal{K}$ for $l = 0, 1$, we define a binary relation E as follows: $(\bar{a}_0, M_0, N_0)E(\bar{a}_1, M_1, N_1)$ iff $M_0 = M_1$ and there exists $N \in \mathcal{K}$ and elementary mappings f_0, f_1 such that $f_l : N_l \to N$ and $f_l \restriction M = id_M$ for $l = 0, 1$ and $f_0(\bar{a}_0) = f_1(\bar{a}_1)$:



Remark III.2.2. E is an equivalence relation on the class of triples of the form $(\bar{a}, M, N)$ where $M \prec_{\mathcal{K}} N$, $\bar{a} \in N$ and both $M, N \in \mathcal{K}^{am}$. When only $M \in \mathcal{K}^{am}$, E may fail to be transitive, but the transitive closure of E could be used instead.

While it is standard to use the E relation to define types in abstract elementary classes, we will discuss and make use of stronger relations between triples in section 3.4 of this paper.

Definition III.2.3. Let β be a positive ordinal (can be one).

(1) For $M, N \in \mathcal{K}^{am}$ and $\bar{a} \in {}^{\beta}\mathrm{N}$. The *Galois type of* $\bar{a}$ *in* N *over* M, written $(\bar{a}/M, N)$, is defined to be $(\bar{a}, M, N)/E$.

(2) We abbreviate $(\bar{a}/M, N)$ by $(\bar{a}/M)$.

(3) For $M \in \mathcal{K}^{am}$,

$$\text{ga-S}^{\beta}(M) := \{(\bar{a}/M, N) \mid M \prec N \in \mathcal{K}^{am}_{\|M\|}, \bar{a} \in {}^{\beta}\mathrm{N}\}.$$

We write ga-S(M) for ga-S$^1(M)$.

(4) Let $p := (\bar{a}/M', N)$ for $M \prec_{\mathcal{K}} M'$ we denote by $p \upharpoonright M$ the type $(\bar{a}/M, N)$. The *domain of* p is denoted by $\operatorname{dom} p$ and it is by definition M'.

(5) Let $p = (\bar{a}/M, N)$, suppose that $M \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} N$ and let $\bar{b} \in {}^{\beta}\mathrm{N}'$ we say that $\bar{b}$ *realizes* p iff $(\bar{b}/M, N') = p \upharpoonright M$.

(6) For types p and q, we write $p \leq q$ if $\operatorname{dom}(p) \subseteq \operatorname{dom}(q)$ and there exists $\bar{a}$ realizing p in some N extending $\operatorname{dom}(p)$ such that $(\bar{a}, \operatorname{dom}(p), N) \in q \upharpoonright \operatorname{dom}(p)$.

Definition III.2.4. We say that $\mathcal{K}$ is *β-stable in μ* if for every $M \in \mathcal{K}^{am}_{\mu}$, $|\,\text{ga-S}^{\beta}(M)| = \mu$. The class $\mathcal{K}$ is *Galois stable in μ* iff $\mathcal{K}$ is 1-stable in μ.

Definition III.2.5. We say that $M \in \mathcal{K}$ is *Galois saturated* if for every $N \prec_{\mathcal{K}} M$ of cardinality $< \|M\|$, and every $p \in \text{ga-S}(N)$, we have that M realizes p.

Remark III.2.6. When $\mathcal{K} = \operatorname{Mod}(T)$ for a first-order T, using the compactness theorem one can show (Theorem 2.2.3 of [Gr1]) that for $M \in \mathcal{K}$, the model M is Galois saturated iff M is saturated in the first-order sense.

It is interesting to mention

Theorem III.2.7 (Shelah [Sh 300]). *Let $\lambda > LS(\mathcal{K})$. Suppose that $\mathcal{K}$ has the amalgamation property and $N \in \mathcal{K}_{\lambda}$. The following are equivalent*

(1) N is Galois staurated.

(2) N is model-homogenous. I.e. if $M \prec_{\mathcal{K}} N$ and $M' \succ M$ of cardinality less than λ then there exists a $\mathcal{K}$-embedding over M from M' into N.

Unfortunately [Sh 300] has an incomplete skeleton of a proof, a complete and correct proof appeared in [Sh 576]. See also [Gr1].

In first order logic, it is natural to consider saturated models for a stable theory. In this context, saturated models are model homogeneous and hence unique. In abstract elementary classes, the existence of saturated models is often difficult to derive without the amalgamation property. To combat this, Shelah introduced a replacement for saturated models, namely, limit-models (Definition II.2.25), whose existence (Theorem II.4.9) and uniqueness (Theorem II.11.1) we have shown in Chapter II for categorical AECs under some additional assumptions.

When $\mathcal{K} = \mathrm{Mod}(T)$ for a first-order and stable T then automatically (by Theorem III.3.12 of [Shc]):

$$M \in \mathcal{K}_\mu \text{ is saturated } \implies M \text{ is } (\mu,\sigma)\text{-limit for all } \sigma < \mu^+ \text{ of cofinality } \geq \kappa(T).$$

When T is countable, stable but not superstable then the saturated model of cardinality μ is $(\mu, \aleph_1)$-limit but not $(\mu, \aleph_0)$-limit.

We have mentioned in Chapter II that the existence of universal extensions follows from categoricity and GCH (see Theorem II.2.21). However, all that is needed for the existence of universal extensions is stability:

Claim III.2.8 (Claim 1.14.1 from [Sh 600]). *Suppose $\mathcal{K}$ is an abstract elementary class with the amalgamation property. If $\mathcal{K}$ is Galois stable in μ, then for every $M \in \mathcal{K}_\mu$, there exists $M' \in \mathcal{K}_\mu$ such that M' is universal over M. Moreover M' can be chosen to be a (μ, σ)-limit over M for any $\sigma < \mu^+$.*

The existence of limit models in stable AECs easily follows from Claim III.2.8 and the amalgamation property. While the uniqueness of limit models is unknown in stable AECs

3.3 Existence of Indiscernibles

Assumption III.3.1. *For the remainder of this chapter, we will fix $\mathcal{K}$, an abstract elementary class with the amalgamation property.*

Remark III.3.2. The focus of this paper are classes with the amalgamation property. Several of the proofs in this section can be adjusted to the context of abstract elementary classes with density of amalgamation bases as in [ShVi] and Chapter II.

The most obvious attempt to generalize Shelah's argument from Lemma I.2.5 of [Shc]for the existence of indiscernibles in first order model theory does not apply since the notion of type cannot be identified with a set of first order formulas. Moreover, there is no natural notion of a type over an arbitrary set in the context of abstract elementary classes. However we do have a notin of non-splitting at our disposal. Recall Shelah's definition of non-splitting from Chapter II:

Definition III.3.3. A type $p \in S^\beta(N)$ *μ-splits* over $M \prec_\mathcal{K} N$ if and only if $\|M\| \leq \mu$, there exist $N_1, N_2 \in \mathcal{K}_{\leq\mu}$ and h, a $\mathcal{K}$-embedding such that $M \prec_\mathcal{K} N_l \prec_\mathcal{K} N$ for $l = 1, 2$ and $h : N_1 \to N_2$ such that $h \restriction M = id_M$ and $p \restriction N_2 \neq h(p \restriction N_1)$.

Notice that non splitting is monotonic: I.e. If $p \in$ ga-S(N) does not split over M (for some $M \prec_\mathcal{K} N$) then p does not split over M' for every $M \prec_\mathcal{K} M' \prec_\mathcal{K} N$. Similarly to $\kappa(T)$ when T is first-order the following is a natural cardinal invariant of $\mathcal{K}$:

Definition III.3.4. Let $\beta > 0$. We define an invariant $\kappa^\beta_\mu(\mathcal{K})$ to be the minimal

κ such that for every $\langle M_i \in \mathcal{K}_\mu \mid i \leq \kappa \rangle$ which satisfies

(1) $\kappa = \text{cf}(\kappa) < \mu^+$,

(2) $\langle M_i \mid i \leq \kappa \rangle$ is $\prec_{\mathcal{K}}$-increasing and continuous and

(3) for every $i < \kappa$, M_{i+1} is a (μ, θ)-limit over M_i for some $\theta < \mu^+$,

and for every $p \in \text{ga-S}^\beta(M_\kappa)$, there exists $i < \kappa$ such that p does not μ-split over M_i. If no such κ exists, we say $\kappa^\beta_\mu(\mathcal{K}) = \infty$.

Notice that Theorem II.7.2 states that categorical abstract elementary classes under Assumption II.1.1 satisfy $\kappa^1_\mu(\mathcal{K}) \leq \omega$, for various μ.

A slight modification of the argument of Claim 3.3 from [Sh 394] can be used to prove a related result using the weaker assumption of Galois-stability only:

Theorem III.3.5. *Let $\beta > 0$. Suppose that $\mathcal{K}$ is β-stable in μ. For every $p \in \text{ga-S}^\beta(N)$ there exists $M \prec_{\mathcal{K}} N$ of cardinality μ such that p does not μ-split over M. Thus $\kappa^\beta_\mu(\mathcal{K}) \leq \mu$.*

For the sake of completness an argument for Theorem III.3.5 is included:

Proof. Suppose $N \succ_{\mathcal{K}} M$, $\bar{a} \in {}^\beta\text{N}$ such that $p = (\bar{a}/M, N)$ and p splits over N_0, for every $N_0 \prec_{\mathcal{K}} M$ of cardinality λ.

Let $\chi := \min\{\chi \mid 2^\chi > \lambda\}$. Notice that $\chi \leq \lambda$ and $2^{<\chi} \leq \lambda$.

We'll define $\{M_\alpha \prec M \mid \alpha < \chi\} \subseteq \mathcal{K}_\lambda$ increasing and continuous $\prec_{\mathcal{K}}$-chain which will be used to construct $M^*_\chi \in \mathcal{K}_\lambda$ such that

$$|\text{ga-S}^\beta(M^*_\chi)| \geq 2^\chi > \lambda \text{ obtaining a contradiction to } \lambda\text{-stability.}$$

Pick $M_0 \prec M$ any model of cardinality λ.

For $\alpha = \beta + 1$; since p splits over M_β there are $N_{\beta,\ell} \prec_{\mathcal{K}} M$ of cardinality λ for $\ell = 1, 2$ and there is $h_\beta : N_{\beta,1} \cong_{M_\beta} N_{\beta,2}$ such that
$h_\beta(p \restriction N_{\beta,1}) \neq p \restriction N_{\beta,2}$. Pick $M_\beta \prec_{\mathcal{K}} M$ of cardinality λ containing the set $|N_{\beta,1}| \cup |N_{\beta,2}|$.

Now for $\alpha < \chi$ define $M^*_\alpha \in \mathcal{K}_\lambda$ and for $\eta \in {}^{\alpha}2$ define a $\mathcal{K}$-embedding h_η such that

(1) $\beta < \alpha \implies M^*_\beta \prec_{\mathcal{K}} M^*_\alpha$,

(2) for α limit let $M^*_\alpha = \bigcup_{\beta<\alpha} M^*_\beta$,

(3) $\beta < \alpha \wedge \eta \in {}^{\alpha}2 \implies h_{\eta\restriction\beta} \subseteq h_\eta$,

(4) $\eta \in {}^{\alpha}2 \implies h_\eta : M_\alpha \overset{\mathcal{K}}{\hookrightarrow} M^*_\alpha$ and

(5) $\alpha = \beta + 1 \wedge \eta \in {}^{\alpha}2 \implies h_{\eta\hat{}0}(N_{\beta,1}) = h_{\eta\hat{}1}(N_{\beta,2})$.

The construction is possible by using the λ-amalgamation property at $\alpha = \beta+1$ several times. Given $\eta \in {}^{\beta}2$ let N^* be of cardinality λ and f_0 be such that the diagram

$$\begin{array}{ccc} M_{\beta+1} & \xrightarrow{f_0} & N^* \\ \uparrow & & \uparrow \\ M_\beta & \xrightarrow[h_\eta]{} & M^*_\beta \end{array}$$

commutes. Denote by N_2 the model $f_0(N_{\beta,2})$. Since $h_\beta : N_{\beta,1} \cong_{M_\beta} N_{\beta,2}$ there is a $\mathcal{K}$-mapping g fixing M_β such that $g(N_{\beta,1}) = N_2$. Using the amalgamation property now pick $N^{**} \in \mathcal{K}_\lambda$ and a mapping f_1 such that the diagram

$$\begin{array}{ccc} M_{\beta+1} & \xrightarrow{f_1} & N^{**} \\ \uparrow & & \uparrow \\ N_{\beta,1} & \xrightarrow[g]{} & N_2 \\ \uparrow & & \uparrow \\ M_\beta & \xrightarrow[h_\eta]{} & M^*_\beta \end{array}$$

Finally apply the amalgamation property to find $M^*_{\beta+1} \in \mathcal{K}_\lambda$ and mappings e_0, e_1 such that

$$\begin{array}{ccc} N^{**} & \xrightarrow{e_1} & M^*_{\beta+1} \\ \uparrow & & \uparrow e_0 \\ M^*_\beta & \longrightarrow & N^* \end{array}$$

commutes. After renaming some of the elements of $M^*_{\beta+1}$ and changing e_1 we may assume that $e_0 =_{N^*}$.

Let $h_{\eta\hat{}0} := f_0$ and $h_{\eta\hat{}1} := e_1 \circ f_1$.

Now for $\eta \in {}^{\chi}2$ let

$$M^*_\chi := \bigcup_{\alpha<\chi} M^*_\alpha \quad \text{and} \quad H_\eta := \bigcup_{\alpha<\chi} h_{\eta\restriction\alpha}.$$

Take $N^*_\eta \succ_{\mathcal{K}} M^*_\chi$ from $\mathcal{K}_\lambda$, an amalgam of N and M^*_χ over M_χ such that

$$\begin{array}{ccc} N & \xrightarrow{H_\eta} & N^*_\eta \\ \uparrow & & \uparrow \\ M_\chi & \xrightarrow[h_\eta]{} & M^*_\chi \end{array}$$

commutes.

Notice that

$$\eta \neq \nu \in {}^{\chi}2 \implies (H_\eta(\bar{a})/M^*_\chi, N^*_\eta) \neq (H_\nu(\bar{a})/M^*_\chi, N^*_\nu).$$

Thus $|\,\text{ga-S}(M^*_\chi)| \geq 2^\chi > \lambda$. $\dashv$

In Theorem III.5.6 below we present an improvement of Theorem III.3.5 for tame AECs: In case $\mathcal{K}$ is β-stable in μ for some μ above its Hanf number then $\kappa^\beta_\mu(\mathcal{K})$ is bounded by the Hanf number. Notice that the bound does not depend on μ.

The following is a new Galois-theoretic notion of indiscernible sequence.

Definition III.3.6. (1) $\langle \bar{a}_i \mid i < i^* \rangle$ is a *Galois indiscernible sequence over* M iff for every $i_1 < \cdots < i_n < i^*$ and every $j_1 < \cdots < j_n < i^*$, $(\bar{a}_{i_1} \ldots \bar{a}_{i_n}/M) = (\bar{a}_{j_1} \ldots \bar{a}_{j_n}/M)$.

(2) $\langle \bar{a}_i \mid i < i^* \rangle$ is a *Galois-indiscernible sequence over* A iff for every $i_1 < \cdots < i_n < i^*$ and every $j_1 < \cdots < j_n < i^*$, there exists $M_i, M_j, M^* \in \mathcal{K}$ and $\prec_{\mathcal{K}}$-mappings f_i, f_j such that

(a) $A \subseteq M_i, M_j$;

(b) $f_l : M_l \rightarrow M^*$, for $l = i, j$;

(c) $f_i(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}) = f_j(\bar{a}_{j_0}, \ldots, \bar{a}_{j_n})$ and

(d) and $f_i \restriction A = f_j \restriction A = id_A$.

Remark III.3.7. This is on the surface a weaker notion of indiscernible sequence than is presented in [Sh 394]. However, this definition coincides with the first order definition. Additionally, it is suspected that, under some reasonable assumptions, this definition and the definition in [Sh 394] are equivalent.

The following lemma provides us with sufficient conditions to find an indiscernible sequence.

Lemma III.3.8. *Let* $\mu \geq LS(\mathcal{K})$*,* κ, λ *be ordinals and* β *a positive ordinal. Suppose that* $\langle M_i \mid i < \lambda \rangle$ *and* $\langle \bar{a}_i \mid i < \lambda \rangle$ *satisfy*

(1) $\langle M_i \in \mathcal{K}_\mu \mid i < \lambda \rangle$ *are* $\preceq_{\mathcal{K}}$*-increasing;*

(2) M_{i+1} *is a* (μ, κ)*-limit over* M_i*;*

(3) $\bar{a}_i \in {}^{\beta}\mathrm{M}_{i+1}$*;*

(4) $p_i := (\bar{a}_i/M_i, M_{i+1})$ *does not* μ*-split over* M_0 *and*

(5) for $i < j < \lambda$*,* $p_i \leq p_j$*.*

Then, $\langle \bar{a}_i \mid i < \lambda \rangle$ *is a Galois-indiscernible sequence over* M_0*.*

Definition III.3.9. A sequence $\langle \bar{a}_i, M_i \mid i < \lambda \rangle$ satisfying conditions $(1) - (6)$ of Lemma III.3.8 is called a *Morley sequence*.

Remark III.3.10. While the statement of the lemma is similar to Shelah's Lemma I.2.5 in [Shc], the proof differs, since types are not sets of formulas.

Proof. We prove that for $i_0 < \cdots < i_n < \lambda$ and $j_0 < \cdots < j_n < \lambda$, $(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/M_0, M_{i_{n+1}}) = (\bar{a}_{j_0}, \ldots, \bar{a}_{j_n}/M_0, M_{j_{n+1}})$ by induction on $n < \omega$.

$n = 0$: Let $i_0, j_0 < \lambda$ be given. Condition 5, gives us

$$(\bar{a}_{i_0}/M_0, M_{i_0+1}) = (\bar{a}_{j_0}/M_0, M_{j_0+1}).$$

$n > 0$: Suppose that the claim holds for all increasing sequences $\bar{i}$ and $\bar{j} \in \lambda$ of length n. Let $i_0 < \cdots < i_n < \lambda$ and $j_0 < \cdots < j_n < \lambda$ be given. Without loss of generality, $i_n \leq j_n$. Define $M^* := M_1$. From condition 2 and uniqueness of (μ, ω)-limits, we can find a $\prec_{\mathcal{K}}$-isomorphism, $g : M_{j_n} \rightarrow M_{i_n}$ such that $g \restriction M_0 = id_{M_0}$. Moreover we can extend g to $g : M_{j_n+1} \rightarrow M_{i_n+1}$. Denote by $\bar{b}_{j_l} := g(\bar{a}_{j_l})$ for $l = 0, \ldots, n$. Notice that $b_{j_l} \in M_{i_n}$ for $l < n$. Since $(\bar{b}_{j_0}, \ldots, \bar{b}_{j_n}/M_0, M_{i_n+1}) = (\bar{a}_{j_0}, \ldots, \bar{a}_{j_n}/M_0, M_{j_n+1})$ it suffices to prove that $(\bar{b}_{j_0}, \ldots, \bar{b}_{j_n}/M_0, M_{i_n+1}) = (\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/M_0, M_{i_n+1})$.

Also notice that the $\prec_{\mathcal{K}}$-mapping preserves some properties of p_j. Namely, since p_j does not μ-split over M_0, $g(p_j \restriction M_{j_n}) = p_j \restriction M_{i_n}$.
Thus, $(\bar{b}_{j_n}/M_{i_n}, M_{i_n+1}) = (\bar{a}_{j_n}/M_{i_n}, M_{i_n+1})$. In particular we have that $(\bar{b}_{j_n}/M_{i_n}, M_{i_n+1})$ does not μ-split over M_0.

By the induction hypothesis

$$(\bar{b}_{j_0}, \ldots, \bar{b}_{j_{n-1}}/M_0, M_{i_n}) = (\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}/M_0, M_{i_n}).$$

Thus we can find $h_i : M_{i_n+1} \to M^*$ and $h_j : M_{i_n+1} \to M^*$ such that $h_i(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}) = h_j(\bar{b}_{j_0}, \ldots, \bar{b}_{j_{n-1}})$. Let us abbreviate $\bar{b}_{j_0}, \ldots, \bar{b}_{j_{n-1}}$ by $\bar{b}_{\bar{j}}$. Similarly we will write $\bar{a}_{\bar{i}}$ for $\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}$.

By appealing to condition 4, we derive several equalities that will be useful in the latter portion of the proof. Since p_j does not μ-split over M_0, we have that $p_j \restriction h_j(M_{i_n}) = h_j(p_j \restriction M_{i_n})$, rewritten as

$$(*) \quad (\bar{b}_{j_n}/h_j(M_{i_n}), M_{i_n+1}) = (h_j(\bar{b}_{j_n})/h_j(M_{i_n}), M^*).$$

Similarly as p_i does not μ-split over M_0, we get $p_i \restriction h_j(M_{i_n}) = h_j(p_i \restriction M_{i_n})$ and $p_i \restriction h_i(M_{i_n}) = h_i(p_i \restriction M_{i_n})$. These equalities translate to

$$(**)_j \quad (\bar{a}_{i_n}/h_j(M_{i_n}), M_{i_n+1}) = (h_j(\bar{a}_{i_n})/h_j(M_{i_n}), M^*) \text{ and}$$

$$(**)_i \quad (\bar{a}_{i_n}/h_i(M_{i_n}), M_{i_n+1}) = (h_i(\bar{a}_{i_n})/h_i(M_{i_n}), M^*), \text{ respectively.}$$

Finally, from condition 5., notice that

$$(***) \quad (\bar{a}_{i_n}/M_{i_n}, M_{i_n+1}) = (\bar{b}_{j_n}/M_{i_n}, M_{i_n+1}).$$

Applying h_j to $(***)$ yields

$$(\dagger) \quad (h_j(\bar{b}_{j_n})/h_j(M_{i_n}), M^*) = (h_j(\bar{a}_{i_n})/h_j(M_{i_n}), M^*).$$

Since $h_i(\bar{a}_{\bar{i}}) = h_j(\bar{b}_{\bar{j}}) \in h_j(M_{i_n})$, we can draw from $(\dagger)$ the following:

$$(1) \quad (h_j(\bar{b}_{j_n})\hat{\ } h_j(\bar{b}_{\bar{j}})/M_0, M^*) = (h_j(\bar{a}_{j_n})\hat{\ } h_i(\bar{a}_{\bar{i}})/M_0, M^*).$$

Equality $(**)_i$ allows us to see

$$(2) \quad (\bar{a}_{i_n}\hat{\ } h_i(\bar{a}_{\bar{i}})/M_0, M^*) = (h_i(\bar{a}_{i_n})\hat{\ } h_i(\bar{a}_{\bar{i}})/M_0, M^*).$$

Since $(h_j(\bar{a}_{i_n})/h_j(M_{i_n}), M^*) = (\bar{a}_{i_n}/h_j(M_{i_n}), M_{i_n+1})$ (equality $(**)_j$)) and $h_i(\bar{a}_{\bar{i}}) = h_j(\bar{b}_{\bar{j}}) \in h_j(M_{i_n})$, we get that

$$(3) \quad (h_j(\bar{a}_{i_n})\hat{\ } h_i(\bar{a}_{\bar{i}})/M_0, M^*) = (\bar{a}_{i_n}\hat{\ } h_i(\bar{a}_{\bar{i}})/M_0, M^*).$$

Combining equalities (1), (2) and (3), we get

$$(\dagger\dagger) \quad (h_i(\bar{a}_{\bar{i}})\hat{\ } h_i(\bar{a}_{i_n})/M_0, M^*) = (h_j(\bar{b}_{\bar{j}})\hat{\ } h_j(\bar{b}_{j_n})/M_0, M^*).$$

Recall that $h_i \restriction M_0 = h_j \restriction M_0 = id_{M_0}$. Thus $(\dagger\dagger)$, witnesses that

$$(\bar{a}_{i_0}, \dots, \bar{a}_{i_n}/M_0, M_{i_n+1}) = (\bar{b}_{j_0}, \dots, \bar{b}_{j_n}/M_0, M_{i_n+1}).$$

$\dashv$

3.4 Tame Abstract Elementary Classes

By Lindström's Theorem, one obvious feature of non-elementary abstract elementary classes is the absence of the compactness theorem. A method of combating this is to view types as equivalences classes of triples (Definition III.2.3) instead of sets of formulas. While this notion of type has led to several profound results in the study of abstract elementary classes, a stronger equivalence relation (denoted E_μ) is eventually utilized in various partial solutions to Shelah's Categoricity Conjecture (see [Sh 394] and [Sh 576]).

Shelah identified E_μ as an interesting relation in [Sh 394]. Here we recall the defintion.

Definition III.4.1. Triples $(\bar{a}_1, M, N_1)$ and $(\bar{a}_2, M, N_2)$ are said to be E_μ-*related* provided that for every $M' \prec_{\mathcal{K}} M$ with $M' \in \mathcal{K}_{<\mu}$,

$$(\bar{a}_1, M', N_1) E (\bar{a}_2, M', N_2).$$

Notice that in first order logic, the finite character of consistency implies that two types are equal if and only if they are E_ω-related.

In Main Claim 9.3 of [Sh 394], Shelah ultimately proves that, under categoricity in some $\lambda > Hanf(\mathcal{K})$ and under the assumption that $\mathcal{K}$ has the amalgamation property, for types over saturated models, E-equivalence is the same as E_μ equivalence for some $\mu < Hanf(\mathcal{K})$.

We now define a context for abstract elementary classes where consistency has small character.

Definition III.4.2. Let χ be a cardinal number. We say the abstract elementary class $\mathcal{K}$ with the amalgamation property is *χ-tame* provided that for types, E-equivalence is the same as the E_χ relation. In other words, for $M \in \mathcal{K}_{>Hanf(\mathcal{K})}$, $p \neq q \in$ ga-S(M) implies existence of $N \prec_{\mathcal{K}} M$ of cardinality χ such that $p \restriction N \neq q \restriction N$.

$\mathcal{K}$ *is tame* iff there exists such that $\mathcal{K}$ is χ-tame for some $\chi < Hanf(\mathcal{K})$

Remark III.4.3. We actually only use that E-equivalence is the same as E_χ-equivalence for types over limit models.

Notice that if $\mathcal{K}$ is a finite diagram (i.e. we have amalgamation not only all models but also over subsets of models) then it is a tame AEC.

There are tame AECs with amalgamation which are not finite diagrams. In fact Leo Marcus in [Ma] constructed an $L_{\omega_1,\omega}$ sentence which is categorical in every cardinal but does not have an uncountable sequentially homogeneous model. Lately Boris Zilber found a mathematically more natural example [Zi].

While we are convinced that there are examples of arbitrary level of tameness at the moment we don't don't any.

Question III.4.4. *For* $\mu_1 < \mu_2 < \beth_{\omega_1}$, *find an AEC which is* μ_2*-tame but not* μ_1*-tame.*

In fact we suspect that the question is easy to answer.

3.5 The order property

The order property, defined next, is an analog of the first order definition of order property using formulas. The order property for non-elementary classes was introduced by Shelah in [Sh 394].

Definition III.5.1. $\mathcal{K}$ is said to have the κ*-order property* provided that for every α, there exists $\langle \bar{d}_i \mid i < \alpha \rangle$ and where $\bar{d}_i \in {}^{\kappa}\mathfrak{C}$ such that if $i_0 < j_0 < \alpha$ and $i_1 < j_1 < \alpha$,

$$(*) \text{ then for no } f \in Aut(\mathfrak{C}) \text{ do we have } f(\bar{d}_{i_0}{}^\frown\bar{d}_{j_0}) = \bar{d}_{j_1}{}^\frown\bar{d}_{i_1}.$$

Remark III.5.2 (Trivial monotonicity). Notice that for $\kappa_1 < \kappa_2$ if a class has the κ_1-order property then it has the κ_2-order property.

Claim III.5.3 (Claim 4.6.3 of [Sh 394]). *We may replace the phrase* every α *in Definition III.5.1 with* every $\alpha < \beth_{(2^{\kappa+LS(\mathcal{K})})^+}$ *and get an equivalent definition.*

Theorem III.5.4 (Claim 4.8.2 of [Sh 394]). *If* $\mathcal{K}$ *has the* κ*-order property and* $\mu \geq \kappa$*, then for some* $M \in \mathcal{K}_\mu$ *we have that* $|\text{ga-S}^{\kappa}(M)/E_\kappa| \geq \mu^+$*. Moreover, we can conclude that* $\mathcal{K}$ *is not Galois stable in* μ*.*

Question III.5.5. *Can we get a version of the stability spectrum theorem for tame stable classes?*

The following is a generaliztion of a old theorem of Shelah from [Sh3] (it is Theorem 4.17 in [GrLe2])

Theorem III.5.6. *Let $\beta > 0$. Suppose that $\mathcal{K}$ is a κ-tame abstract elementary class. If $\mathcal{K}$ is β-stable in μ with $\beth_{(2^{\kappa+LS(\mathcal{K})})^+} \leq \mu$, then $\kappa^\beta_\chi(\mathcal{K}) < \beth_{(2^{\kappa+LS(\mathcal{K})})^+}$.*

Proof. Let $\chi := \beth_{(2^{\kappa+LS(\mathcal{K})})^+}$. Suppose that the conclusion of the theorem does not hold. Let $\langle M_i \in \mathcal{K}_\mu \mid i \leq \chi\rangle$ and $p \in \text{ga-S}^\beta(M_\chi)$ witness the failure. Namely, the following hold:

(1) $\langle M_i \mid i \leq \chi\rangle$ is $\prec_\mathcal{K}$-increasing and continuous,

(2) for every $i < \chi$, M_{i+1} is a (μ, θ)-limit over M_i for some $\theta < \mu^+$ and

(3) for every $i < \mu^+$, p μ-splits over M_i.

For every $i < \chi$ let f_i, N^1_i and N^2_i witness that p μ-splits over M_i. Namely,

$$M_i \prec_\mathcal{K} N^1_i, N^2_i \prec_\mathcal{K} M,$$

$$f_i : N^1_i \cong N^2_i \text{ with } f_i \restriction M_i = id_{M_i}$$

$$\text{and } f_i(p \restriction N^1_i) \neq p \restriction N^2_i.$$

By κ-tameness, there exist B_i and $A_i := f^{-1}_i(B_i)$ of size $< \kappa$ such that

$$f_i(p \restriction A_i) \neq p \restriction B_i.$$

By renumbering our chain of models, we may assume that

(4) $A_i, B_i \subset M_{i+1}$.

Since M_{i+1} is a limit model over M_i, we can additionally conclude that

(5) $\bar{c}_i \in M_{i+1}$ realizes $p \restriction M_i$.

For each $i < \mu$, let $\bar{d}_i := A_i \hat{} B_i \hat{} \bar{c}_i$.

Claim III.5.7. *$\langle \bar{d}_i \mid i < \chi \rangle$ witnesses the κ-order property.*

Proof. Suppose for the sake of contradiction that there exist $g \in Aut(\mathfrak{C})$, $i_0 < j_0 < \chi$ and $i_1 < j_1 < \chi$ such that

$$g(\bar{d}_{i_0} \hat{} \bar{d}_{j_0}) = \bar{d}_{j_1} \hat{} \bar{d}_{i_1}.$$

Notice that since $i_0 < j_0 < \alpha$ we have that $\bar{c}_{i_0} \in M_{j_0}$. So $f_{j_0}(\bar{c}_{i_0}) = \bar{c}_{i_0}$. Recall that $f_{j_0}(A_{j_0}) = B_{j_0}$. Thus, f_{j_0} witnesses that

$$(*)(\bar{c}_{i_0} \hat{} A_{j_0}/\emptyset) = (\bar{c}_{i_0} \hat{} B_{j_0}/\emptyset).$$

Applying g to $(*)$ we get

$$(**)(\bar{c}_{j_1} \hat{} A_{i_1}/\emptyset) = (\bar{c}_{j_1} \hat{} B_{i_1}/\emptyset).$$

Applying f_{i_1} to the RHS of $(**)$, we notice that

$$(\sharp)(f_{i_1}(\bar{c}_{j_1}) \hat{} B_{i_1}/\emptyset) = (\bar{c}_{j_1} \hat{} B_{i_1}/\emptyset).$$

Because $i_1 < j_1$, we have that $\bar{c}_{j_1}$ realizes $p \restriction M_{i_1}$. Thus, $(\sharp)$ implies

$$(\sharp\sharp) f_{i_1}(p \restriction A_{i_1}) = p \restriction B_{i_1},$$

which contradicts our choice of f_{i_1}, A_{i_1} and B_{i_1}.

$\dashv$

By Claim III.5.3 and Theorem III.5.4, we have that $\mathcal{K}$ is unstable in μ, contradicting our hypothesis.

$\dashv$

3.6 Morley sequences

Hypothesis III.6.1. For the rest of the chapter we make the following assumption: $\mathcal{K}$ is a tame abstract elementary class, has no maximal models and satisfies the amalgamation property.

Theorem III.6.2. *Suppose $\mu \geq \beth_{(2^{Hanf(\mathcal{K})})^+}$. Let $M \in \mathcal{K}_{>\mu}$, $A, I \subset M$ be given such that $|I| \geq \mu^+ > |A|$. If $\mathcal{K}$ is Galois stable in μ, then there exists $J \subset I$ of cardinality μ^+, Galois indiscernible over A. Moreover J can be chosen to be a Morley sequence over A.*

Proof. Fix $\kappa := \text{cf}(\mu)$. Let $\{\bar{a}_i \mid i < \mu^+\} \subseteq I$ be given. Define $\langle M_i \in K_\mu \mid i < \mu^+\rangle$ $\prec_{\mathcal{K}}$-increasing and continuous satisfying

(1) $A \subseteq |M_0|$

(2) M_{i+1} is a (μ, κ)-limit over M_i

(3) $\bar{a}_i \in M_{i+1}$

Let $p_i := (\bar{a}_i/M_i, M_{i+1})$ for every $i < \mu^+$. Define $f : S^{\mu^+}_\kappa \to \mu^+$ by

$$f(i) := \min\{j < \mu^+ \mid p_i \text{ does not } \mu\text{- split over } M_j\}.$$

By Theorem III.5.6, f is regressive. Thus by Fodor's Lemma, there are a stationary set $S \subseteq S^{\mu^+}_\kappa$ and $j_0 \in I$ such that for every $i \in S$,

$$(\dagger) \quad p_i \text{ does not } \mu\text{-split over } M_{j_0}.$$

By stability and the pigeon-hole principle there exists $p^* \in \text{ga-S}(M_{j_0})$ and $S^* \subseteq S$ of cardinality μ^+ such that for every $i \in S^*$, $p^* = p_i \restriction M_{j_0}$. Enumerate and rename S^*. Let $M^* := M_1$. Again, by stability we can find $S^{**} \subset S^*$ of cardinality μ^+ such that for every $i \in S^{**}$, $p^{**} = p_i \restriction M^*$. Enumerate and rename S^{**}.

Subclaim III.6.3. *For $i < j \in S^{**}$, $p_i = p_j \restriction M_i$.*

Proof. Let $0 < i < j \in S^{**}$ be given. Since M_{i+1} and M_{j+1} are (μ, κ)-limits over M_i, there exists an isomorphism $g : M_{j+1} \to M_{i+1}$ such that $g \restriction M_i = id_{M_i}$. Let $\bar{b}_j := g(\bar{a}_j)$. Since the type p_j does not μ-split over M_{j_0}, g cannot witness the splitting. Therefore, it must be the case that $(\bar{b}_j/M_i, M_{i+1}) = p_i \restriction M_i$. Then, it suffices to show that $(\bar{b}_j/M_i, M_{i+1}) = p_i$.

Since $p_i \restriction M_0 = p_j \restriction M_0$, we can find $\prec_{\mathcal{K}}$-mappings witnessing the equality. Furthermore since M^* is universal over M_0, we can find $h_l : M_{l+1} \to M^*$ such that $h_l \restriction M_0 = id_{M_0}$ for $l = i, j$ and $h_i(\bar{a}_i) = h_j(\bar{b}_j)$.

We will use (†) to derive several inequalities. Consider the following possible witness to splitting. Let $N_1 := M_i$ and $N_2 := h_i(M_i)$. Since p_i does not μ-split over M_0, we have that $p_i \restriction N_2 = h_i(p_i \restriction N_1)$, rewritten as

$$(*) \quad (\bar{a}_i/h_i(M_i), M_{i+1}) = (h_i(\bar{a}_i)/h_i(M_i), M^*).$$

Similarly we can conclude that

$$(**) \quad (\bar{b}_j/h_j(M_i), M_{i+1}) = (h_j(\bar{b}_j)/h_j(M_i), M^*).$$

By choice of S^{**}, we know that

$$(***) \quad (\bar{b}_j/M^*) = (\bar{a}_i/M^*).$$

Now let us consider another potential witness of splitting. $N_1^* := h_i(M_i)$ and $N_2^* := h_j(M_i)$ with $H^* := h_j \circ h_i^{-1} : N_1^* \to N_2^*$. Since $p_j \restriction M_i$ does not μ-split over M_0, $p_j \restriction N_2^* = H^*(p_j \restriction N_1^*)$. Thus by $(**)$ we have

$$(\sharp) \quad H^*(p_j \restriction N_1^*) = (h_j(\bar{b}_j)/h_j(M_i), M^*).$$

Now let us translate $H^*(p_j \restriction N_1^*)$. By monotonicity and $(***)$, we have that $p_j \restriction N_1^* = (\bar{b}_j/h_i(M_i), M_{i+1}) = (\bar{a}_i/h_i(M_i), M_{i+1})$. We can then conclude by $(*)$ that $p_j \restriction N_1^* = (h_i(\bar{a}_i)/h_i(M_i), M_{i+1})$. Applying H^* to this equality yields

$$(\sharp\sharp) \quad H^*(p_j \restriction N_1^*) = (h_j(\bar{a}_i)/h_j(M_i), M^*).$$

By combining the equalities from $(\sharp)$ and $(\sharp\sharp)$ and applying h_j^{-1} we get that

$$(\bar{b}_j/M_i, M_{i+1}) = (\bar{a}_i/M_i, M_{i+1}).$$

$\dashv$

Notice that by Subclaim III.6.3 and our choice of S^{**}, $\langle M_i \mid i \in S^{**}\rangle$ and $\langle \bar{a}_i \mid i \in J\rangle$ satisfy the conditions of Lemma III.3.8. Applying Lemma III.3.8, we get that $\langle \bar{a}_i \mid i \in S^{**}\rangle$ is a morley sequence over M_0. In particular, since $A \subset M_0$, we have that $\langle \bar{a}_i \mid i \in S^{**}\rangle$ is a Morley sequence over A.

$\dashv$

3.7 Exercise on Dividing

With the existence of Morley sequences a natural extension is to study the following dependence relation to determine whether or not it satisfies properties such as transitivity, symmetry or extension. Here we derive the existence property.

Definition III.7.1. Let $p \in \text{ga-S}(M)$ and $N \prec_{\mathcal{K}} M$. We say that p *divides over* N iff there are $\bar{a} \in M$ non-algebraic over N and a Morley sequence, $\{\bar{a}_n \mid n < \omega\}$ for the $(\bar{a}/N, M)$ such that for every collection $\{f_n \in Aut_M \mathfrak{C} \mid n < \omega\}$ with $f_n(\bar{a}) = \bar{a}_n$ we have

$$\{f_n(p) \mid n < \omega\} \text{ is inconsistent.}$$

Theorem III.7.2 (Existence). *Suppose that $\mathcal{K}$ is stable in μ and κ-tame for some $\kappa < \mu$. For every $p \in$ ga-S(M) with $M \in \mathcal{K}_{\geq\mu}$ there exists $N \prec_{\mathcal{K}} M$ of cardinality μ such that p does not divide over N.*

Proof. Suppose that p and M form a counter-example. WLOG we may assume that $M = \mathfrak{C}$. Through the proof of Claim 3.3.1 of [Sh 394], in order to contradict stability in μ, it suffices to find N_i, N_i^1, N_i^2, h_i for $i < \mu$ satisfying

(1) $\langle N_i \in \mathcal{K}_\mu \mid i \leq \mu \rangle$ is a $\prec_{\mathcal{K}}$-increasing and continuous sequence of models;

(2) $N_i \prec_{\mathcal{K}} N_i^l \prec_{\mathcal{K}} N_{i+1}$ for $i < \mu$ and $l = 1, 2$;

(3) for $i < \mu$, $h_i : N_i^1 \cong N_i^2$ and $h_i \restriction N_i = id_{N_i}$ and

(4) $p \restriction N_i^2 \neq h_i(p \restriction N_i^1)$.

Suppose that N_i has been defined. Since p divides over every substructure of cardinality μ, we may find $\bar{a}$, $\{\bar{a}_n \mid n < \omega\}$ and $\{f_n \mid n < \omega\}$ witnessing that p divides over N_i. Namely, we have that $\{f_n(p) \mid n < \omega\}$ is inconsistent. Let $n < \omega$ be such that $f_0(p) \neq f_n(p)$. Then $p \neq f_0^{-1} \circ f_n(p)$. By κ-tameness, we can find $N^* \prec_{\mathcal{K}} \mathfrak{C}$ of cardinality μ containing N such that $p \restriction N^* \neq (f_0^{-1} \circ f_n(p)) \restriction N^*$. WLOG $f_0^{-1} \circ f_n \in Aut_N N^*$.

Let $h_i := f_0^{-1} \circ f_n$, $N_i^1 := N^*$ and $N_i^2 := N^*$. Choose $N_{i+1} \prec_{\mathcal{K}} \mathfrak{C}$ to be an extension of N^* of cardinality μ. $\dashv$

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