A SUBSTITUTE FOR SATURATION IN ABSTRACT ELEMENTARY CLASSES

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Abstract. This paper continues work of Shelah and Villaveces from [ShVi]. Shelah and Villaveces work towards a downward categoricity transfer theorem in abstract elementary classes (AEC) with no maximal models under GCH and a version of the weak diamond.

One of the main conjectures in [ShVi] was the uniqueness of limit models. Here we solve this problem under an additional assumption. Suppose that $\mathcal{K}$ is an AEC categorical in some λ above the Hanf number and that for $\mu<\lambda$, $\mathcal{K}^{am}_\mu$ is closed under unions of length $<\mu^+$.

Theorem 0.1 (Uniqueness of Limit Models)**.** *Let θ_1,θ_2 be limit ordinals $<\mu^+$. If M_1 is a (μ,θ_1)-limit model over M and M_2 is a (μ,θ_2)-limit model over M, then M_1 is isomorphic to M_2.*

In order to prove the uniqueness theorem, we develop the theory of Galois-splitting in AECs. In particular we prove the extension property for Galois-splitting in AECs:

Theorem 0.2 (Extension Property for Splitting)**.** *Suppose that* ga-tp(a/M) *does not μ-split over N and that M is universal over N. For every $M'\in\mathcal{K}^{am}_\mu$ with $M\prec_{\mathcal{K}} M'$, we have that there exists $q\in$* ga-S(M') *such that $q\supseteq$* ga-tp(a/M) *and q does not μ-split over N.*

We generalize Theorem 0.2 and prove the $<^c$-extension property for towers. This is also a partial solution to a problem from [ShVi]. In our proof of Theorem 0.1, we develop the concept of full towers. We show that the union of full towers is full.

1. Introduction

Shelah's paper, [Sh 702] is based on a series of lectures given at Rutgers University. In the lectures, Shelah elaborates on open problems in model theory which he has attempted but which have not yet been solved. There Shelah refers to the subject of Section 13, "Classification of Non-elementary Classes," as the major problem of model theory. He points out that one of the main steps in classifying non-elementary classes is the development of stability theory. In first order logic, solutions to Łŏs' Conjecture produced machinery that advanced the study of stability theory. It is natural, then, to

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consider a generalization of this conjecture as a test question for a proposed stability theory for AECs:

Conjecture 1.1 ([Sh 702])**.** *If $\mathcal{K}$ is an abstract elementary class that is categorical in some $\lambda > Hanf(\mathcal{K})$, then $\mathcal{K}$ is categorical in every $\mu > Hanf(\mathcal{K})$.*

Despite the existence of over 500 published pages of partial results towards this conjecture, it remains very open. Since the mid-eighties, model theorists have approached Shelah's conjecture from two different directions. Shelah, M. Makkai and O. Kolman attacked the conjecture with set theoretic assumptions (see [MaSh], [KoSh] and [Sh 472]). On the other hand, Shelah also looked at the conjecture under additional model theoretic assumptions in [Sh 394] and [Sh 600]. More recent work of Shelah and A. Villaveces [ShVi] profits from both model theoretic and set theoretic assumptions, however these assumptions are weaker than the hypotheses made in [MaSh], [KoSh], [Sh 472], [Sh 394], and [Sh 600]. A main feature of their context is that they work in AECs where the amalgamation property is not known to hold. This paper focuses on resolving problems from [ShVi]. Here we recall the context of [ShVi] (Assumptions 1.2.(1) through 1.2.(5)).

Assumption 1.2. We make the following assumptions for the remainder of the paper:

(1) $\mathcal{K}$ is an abstract elementary class,
(2) $\mathcal{K}$ has no maximal models,
(3) $\mathcal{K}$ is categorical in some $\lambda > LS(\mathcal{K})$,
(4) GCH holds and
(5) $\Phi_{\mu^+}(S^{\mu^+}_\theta)$ holds for every cardinal $\mu < \lambda$ and every regular θ with $\theta < \mu^+$.

Assumption 1.2.(5) is not explicitly made in [ShVi], but it is implicit in Hypothesis 1.3.8 from [ShVi]. We provide a complete proof of the theorem which uses Hypothesis 1.3.8 (see Theorem 4.7) and give an exposition of the strength of Assumption 1.2.5 in Section 4.

In 1985 Rami Grossberg made the following conjecture:

Conjecture 1.3. *If $\mathcal{K}$ is an AEC categorical above the Hanf number, then every $M \in \mathcal{K}$ is an amalgamation base.*

This conjecture encouraged Shelah to produce a partial solution to the categoricity conjecture under the assumption that every model $M \in \mathcal{K}$ is an amalgamation base [Sh 394]. This result directs future work towards the categoricity conjecture to solving Conjecture 1.3. The underlying goal of [ShVi] was to make progress towards Conjecture 1.3 under Assumption 1.2. Not knowing that every model is an amalgamation base presents several obstacles in applying known notions and techniques. For instance, there may exist some models over which we cannot even define the most basic notion of a type.

One approach to Conjecture 1.3 is to see if arguments from [KoSh] can be carried out in this more general context. Shelah and Kolman prove Conjecture 1.3 for $L_{\kappa,\omega}$ theories where κ is a measurable cardinal. They first introduce limit models as a substitute for saturated models, and then prove the uniqueness of limit models. A major objective of [ShVi] was to show the uniqueness of limit models:

Conjecture 1.4 (Uniqueness of Limit Models)**.** *Suppose Assumption 1.2 holds. For $\theta_1, \theta_2 < \mu^+ < \lambda$, if M_1 and M_2 and (μ, θ_1)-, (μ, θ_2)-limit models over M, respectively, then M_1 is isomorphic to M_2.*

While limit models were used to prove that every model is an amalgamation base in [KoSh], limit models played a *behind-the-scenes* role in Shelah's downward solution to the categoricity conjecture in [Sh 394]. Furthermore, there is evidence that the uniqueness of limit models provides a basis for the development of a notion of non-forking for abstract elementary classes. Such a notion could be used in the development of a stability theory for AECs.

In the Fall of 1999, I identified a gap in Shelah and Villaveces' proof of uniqueness of limit models. As of the Fall of 2001, Shelah and Villaveces could not resolve the problem.

This paper includes an exposition of selected results from [ShVi] and includes a partial solution to the uniqueness of limit models (see Theorem 2.28). In order to prove Theorem 2.28, we prove several facts about splitting and full towers which stand alone.

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2. Background

Definition 2.1. $\mathcal{K}$ is an *abstract elementary class (AEC)* iff $\mathcal{K}$ is a class of models for some vocabulary τ and is equipped with a binary relation, $\preceq_{\mathcal{K}}$ satisfying the following:

(1) Closure under isomorphisms.
(2) $\preceq_{\mathcal{K}}$ refines the submodel relation.
(3) $\preceq_{\mathcal{K}}$ is a partial order on $\mathcal{K}$.
(4) If $\langle M_i \mid i < \delta\rangle$ is a $\prec_{\mathcal{K}}$-increasing and chain of models in $\mathcal{K}$
   (a) $\bigcup_{i<\delta} M_i \in \mathcal{K}$,
   (b) for every $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$ and
   (c) if $M_i \prec_{\mathcal{K}} N$ for every $i < \delta$, then $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} N$.
(5) If $M_0, M_1 \preceq_{\mathcal{K}} N$ and M_0 is a submodel of M_1, then $M_0 \preceq_{\mathcal{K}} M_1$.
(6) (Downard Löwenheim-Skolem Axiom) There is a Löwenheim-Skolem number of $\mathcal{K}$, denoted $LS(\mathcal{K})$ which is the minimal κ such that for

every $N \in \mathcal{K}$ and every $A \subset N$, there exists M with $A \subseteq M \prec_{\mathcal{K}} N$ of cardinality $\kappa + |A|$.

Notation 2.2. If λ is a cardinal and $\mathcal{K}$ is an abstract elementary class, $\mathcal{K}_\lambda$ is the collection of elements of $\mathcal{K}$ with cardinality λ.

Definition 2.3. For models M, N in an AEC, $\mathcal{K}$, the mapping $f : M \to N$ is an $\prec_{\mathcal{K}}$-*embedding* iff f is an injective $L(\mathcal{K})$-homomorphism and $f[M] \preceq_{\mathcal{K}} N$.

Using the axioms of AEC, one can show that Axiom 4 has an alternative formulation (see [Sh 88] or Chapter 13 of [Gr]):

Proposition 2.4 (P.M. Cohn 1965)**.** *Let $(I, \leq)$ be a directed set. If $\langle M_t \mid t \in I\rangle$ and $\{h_{t,s} \mid t \leq s \in I\}$ are such that*

1. *for $t \in I$, $M_t \in \mathcal{K}$*
2. *for $t \leq s \in I$, $h_{t,s} : M_t \to M_s$ is a $\prec_{\mathcal{K}}$-embedding and*
3. *for $t_1 \leq t_2 \leq t_3 \in I$, $h_{t_1,t_3} = h_{t_2,t_3} \circ h_{t_1,t_2}$ and $h_{t,t} = id_{M_t}$,*

then, whenever $s = \lim_{t\in I} t$, there exist $M_s \in \mathcal{K}$ and $\prec_{\mathcal{K}}$-mappings $\{h_{t,s} \mid t \in I\}$ such that

$$h_{t,s} : M_t \to M_s \text{ and}$$
$$\text{for } t_1 \leq t_2 \leq s, h_{t_1,s} = h_{t_2,s} \circ h_{t_1,t_2} \text{ and } h_{s,s} = id_{M_s}.$$

Definition 2.5. A partially ordered set $(I, \leq)$ is *directed* iff for every $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$.

We will make use of a related lemma about directed limits:

Proposition 2.6. *$\mathcal{K}^{\prec_{\mathcal{K}}} := \{(N, M) \mid M, N \in \mathcal{K}, M \prec_{\mathcal{K}} N\}$ is an abstract elementary class with $L(\mathcal{K}^{\prec_{\mathcal{K}}}) = L(\mathcal{K}) \bigcup \{P\}$ where P is a unary predicate and $\prec_{\mathcal{K}^{\prec_{\mathcal{K}}}}$ is defined by*

$$(N, M) \prec_{\mathcal{K}^{\prec_{\mathcal{K}}}} (N', M') \Leftrightarrow (N \prec_{\mathcal{K}} N' \text{ and } M \prec_{\mathcal{K}} M').$$

From this proposition and the existence and uniqueness of direct limits for AECs, we have:

Lemma 2.7. *Suppose that $\langle M_t \prec_{\mathcal{K}} N_t \mid t \in I\rangle$ and $\langle f_{t,s} \mid t \leq s \in I\rangle$ is a directed system with $f_{t,s} : N_t \to N_s$ and $f_{t,s} \restriction M_t : M_t \to M_s$. If M^* and N^* are the direct limits of the directed sets $\langle M_t, f_{t,s} \restriction M_t \mid t \leq s \in I\rangle$ and $\langle N_t, f_{t,s} \mid t \leq s \in I\rangle$, respectively, then $M^* \prec_{\mathcal{K}} N^*$ and $f_{t,\sup(I)} \restriction M_t \to M^*$.*

We will use Lemma 2.7 as well as the trivial observation (Claim 2.8) in the proof of the Conjecture 1.4.

Claim 2.8. *If $\langle N_t \mid t < s\rangle$ and $\langle f_{r,t} \mid r < t < s\rangle$ form a directed system and for every $r \leq t < s$ we have that $N_t = N_r = N$ and $f_{r,t} \in Aut(N)$. Then the direct limit $(N_s, \langle f_{t,s} \mid t \leq s\rangle)$ is such that $f_{t,s} : N_t \cong N_s$ for every $t \leq s$. Moreover we can choose $N_s = N$.*

The following gives a characterization of AECs as PC-classes. Theorem 2.10 is often referred to as Shelah's Presentation Theorem.

Definition 2.9. A class $\mathcal{K}$ of structures is called a *$PC-class$* if there exists a language L_1, a first order theory, T_1, in the language, L_1, and a collection of types without parameters, Γ, such that L_1 is an expansion of $L(\mathcal{K})$ and

$$\mathcal{K} = PC(T_1, \Gamma, L) := \{M \upharpoonright L : M \models T_1 \text{ and } M \text{ omits all types from } \Gamma\}.$$

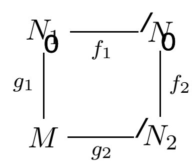
When $|T_1| + |L_1| + |\Gamma| + \aleph_0 = \mu$, we say that $\mathcal{K}$ is PC_μ.

Theorem 2.10 (Lemma 1.8 of [Sh 88] or [Gr])**.** *If $(\mathcal{K}, \prec_{\mathcal{K}})$ is an AEC, then there exists $\mu \leq 2^{LS(\mathcal{K})}$ such that $\mathcal{K}$ is PC_μ.*

In Section 3 we will see that this presentation of AECs as PC-classes allows us to construct Ehrenfuecht-Mostowski models.

Definition 2.11. Let $\mathcal{K}$ be an abstract elementary class.

(1) Let μ,κ_1 and κ_2 be cardinals with $\mu \leq \kappa_1, \kappa_2$. We say that $M \in \mathcal{K}_\mu$ is a (κ_1, κ_2)-*amalgamation base* if for every $N_1 \in \mathcal{K}_{\kappa_1}$ and $N_2 \in \mathcal{K}_{\kappa_2}$ and $g_i : M \to N_i$ for $(i = 1, 2)$, there are $\prec_{\mathcal{K}}$-embeddings f_i, $(i = 1, 2)$ and a model N such that the following diagram commutes:

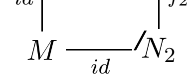


(2) We say that a model $M \in \mathcal{K}_\mu$ is an *amalgamation base* if M is a (μ, μ)-amalgamation base.
(3) We write $\mathcal{K}^{am}$ for the class of amalgamation bases which are in $\mathcal{K}$.
(4) We say $\mathcal{K}$ satisfies the *amalgamation property* iff for every $M \in \mathcal{K}$, M is an amalgamation base.

Remark 2.12. We get an equivalent definition of amalgamation base, if we additionally require that $g_i \upharpoonright M = id_M$ for $i = 1, 2$, in the definition above. See [Gr] for details.

Amalgamation bases are central in the definition of types. Since we are not working in a fixed logic, we will not define types as collections of formulas. Instead, we will define types as equivalence classes with respect to images under $\prec_{\mathcal{K}}$-mappings:

Definition 2.13. For triples $(\bar{a}_l, M_l, N_l)$ where $\bar{a}_l \in N_l$ and $M_l \preceq_{\mathcal{K}} N_l \in \mathcal{K}$ for $l = 0, 1$, we define a binary relation E as follows: $(\bar{a}_0, M_0, N_0)E(\bar{a}_1, M_1, N_1)$ iff $M_0 = M_1$ and there exists $N \in \mathcal{K}$ and $\prec_{\mathcal{K}}$-mappings f_0, f_1 such that $f_l : N_l \to N$ and $f_l \upharpoonright M = id_M$ for $l = 0, 1$ and $f_0(\bar{a}_0) = f_1(\bar{a}_1)$:



Remark 2.14. E is an equivalence relation on the set of triples of the form $(\bar{a}, M, N)$ where $M \preceq_{\mathcal{K}} N$, $\bar{a} \in N$ and $M, N \in \mathcal{K}^{am}_\mu$ for fixed $\mu \geq LS(\mathcal{K})$.

In AEC with the amalgamation property, we are often limited to speak of types only over models. Here we are further restricted to deal with types only over models which are amalgamation bases.

Definition 2.15. Let $\mu \geq LS(\mathcal{K})$ be given.

(1) For $M, N \in \mathcal{K}^{am}_\mu$ and $\bar{a} \in {}^{\omega>}\mathrm{N}$, the *Galois-type of $\bar{a}$ in N over M*, written ga-tp$(\bar{a}/M, N)$, is defined to be $(\bar{a}, M, N)/E$.
(2) For $M \in \mathcal{K}^{am}_\mu$, ga-S$^1(M) := \{$ga-tp$(a/M, N) \mid M \preceq N \in \mathcal{K}^{am}_\mu, a \in N\}$.
(3) We say $p \in$ ga-S(M) is realized in N whenever $M \prec_{\mathcal{K}} N$ and there exist $a \in N$ and $N' \in \mathcal{K}^{am}_\mu$ such that $p = (a, M, N')/E$.

Remark 2.16. We refer to these types as Galois-types to distinguish them from notions of types defined as a collection of formulas.

Proposition 2.17 (see [Gr]). *When $\mathcal{K} = \mathrm{Mod}(T)$ for T a complete first order theory, the above definition of* ga-tp$(a/M, N)$ *coincides with the classical first order defintion where c and a have the same type over M iff for every first order formula $\varphi(x, \bar{b})$ with parameters from M,*

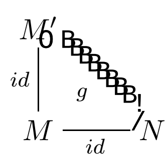
$$\models \varphi(c, \bar{b}) \leftrightarrow \models \varphi(a, \bar{b}).$$

Proof. By Robinson's Consistency Theorem. ⊣

Definition 2.18. We say that $\mathcal{K}$ is *stable in μ* if for every $M \in \mathcal{K}^{am}_\mu$, $|$ga-S$^1(M)| = \mu$.

Fact 2.19 (Fact 2.1.3 of [ShVi]). *Since $\mathcal{K}$ is categorical in λ, for every $\mu < \lambda$, we have that $\mathcal{K}$ is stable in μ.*

Definition 2.20. (1) Let κ be a cardinal. We say N is *κ-universal over M* iff for every $M' \in \mathcal{K}_\kappa$ with $M \prec_{\mathcal{K}} M'$ there exists a $\prec_{\mathcal{K}}$-embedding $g : M' \to N$ such that $g \restriction M = id_M$:



(2) We say N is *universal over M* iff N is $\|M\|$-universal over M.

Universal extensions exist in many stable contexts [Sh 600]. We provide a reference to [ShVi] where a proof is provided from stronger assumptions including categoricity and GCH (see Lemma 2.21).

Lemma 2.21 (Theorem 1.3.1 from [ShVi]). *For every μ with $LS(\mathcal{K}) < \mu < \lambda$, if $M \in \mathcal{K}^{am}_\mu$, then there exists $M' \in \mathcal{K}^{am}_\mu$ such that M' is universal over M.*

Notice that the following proposition asserts that it is unreasonable to prove a stronger existence statement than Lemma 2.21, without having proved the amalgamation property.

Proposition 2.22. *If M' is universal over M, then M is an amalgamation base.*

As mentioned in the introduction, limit models were introduced by Kolman and Shelah in [KoSh]. After proving the uniqueness of limit models in their context, Shelah and Kolman derive the Amalgamation Property. The main goal of this paper is to prove the uniqueness of limit models in the context of [ShVi] under an additional assumption.

Definition 2.23. For $M', M \in \mathcal{K}_\mu$ and σ a limit ordinal with $\sigma < \mu^+$, we say that M' is a (μ, σ)*-limit over* M iff there exists a $\prec_{\mathcal{K}}$-increasing and continuous sequence of models $\langle M_i \in \mathcal{K}_\mu \mid i < \sigma \rangle$ such that

(1) $M \preceq_{\mathcal{K}} M_0$,
(2) $M' = \bigcup_{i<\sigma} M_i$
(3) for $i < \sigma$, M_i is an amalgamation base and
(4) M_{i+1} is universal over M_i.

Remark 2.24. (1) Notice that in Definition 2.23, for $i < \sigma$ and i a limit ordinal, M_i is a (μ, i)-limit model.
(2) Notice that Condition (4) implies Condition (3) of Definition 2.23.

Definition 2.25. We say that M' is a (μ, σ)*-limit* iff there is some $M \in \mathcal{K}$ such that M' is a (μ, σ)-limit over M.

Notation 2.26. (1) For μ a cardinal and σ a limit ordinal with $\sigma < \mu^+$, we write $\mathcal{K}^\sigma_\mu$ for the collection of (μ, σ)-limit models of $\mathcal{K}$.
(2) We define

$$\mathcal{K}^*_\mu := \{M \in \mathcal{K} \mid M \text{ is a } (\mu, \theta)-\text{limit model for some limit ordinal } \theta < \mu^+\}.$$

as the *collection of limit models of* $\mathcal{K}$.

Limit models also exist in certain abstract elementary classes. By repeated applications of Lemma 2.21, the existence of (μ, ω)-limit models can be proved:

Proposition 2.27 (Theorem 1.3.1 from [ShVi])**.** *Let μ be a cardinal such that $\mu < \lambda$. For every $M \in \mathcal{K}^{am}_\mu$, there exists $M' \in \mathcal{K}$ such that $M \prec_{\mathcal{K}} M'$ and M' is a (μ, ω)-limit over M.*

In order to extend this argument further to yield the existence of (μ, σ)-limits for arbitrary limit ordinals $\sigma < \mu^+$, we need to be able to verify that limit models are in fact amalgamation bases. We will examine this in Section 4.

While the existence of certain limit models is relatively easy to derive from the categoricity assumption, the uniqueness of limit models is more difficult. Here we recall two easy uniqueness facts which state that limit models of the same length are isomorphic:

Proposition 2.28 (Fact 1.3.6 from [ShVi])**.** *Let $\mu \geq LS(\mathcal{K})$ and $\sigma < \mu^+$. If M_1 and M_2 are (μ,σ)-limits over M, then there exists an isomorphism $g : M_1 \to M_2$ such that $g \restriction M = id_M$. Moreover if M_1 is a (μ,σ)-limit over M_0; N_1 is a (μ,σ)-limit over N_0 and $g : M_0 \cong N_0$, then there exists a $\prec_{\mathcal{K}}$-mapping, $\hat{g}$, extending g such that $\hat{g} : M_1 \cong N_1$.*

Proposition 2.29 (Fact 1.3.7 from [ShVi])**.** *Let μ be a cardinal and σ a limit ordinal with $\sigma < \mu^+ \leq \lambda$. If M is a (μ,σ)-limit model, then M is a $(\mu, cf(\sigma))$-limit model.*

A more challenging uniqueness question, which was a major goal of [ShVi], is:

Conjecture 2.30. *Suppose $\mathcal{K}$ is categorical in some $\lambda > LS(\mathcal{K})$. Let $\mu \geq LS(\mathcal{K})$ and $\theta_1, \theta_2 < \mu^+ \leq \lambda$ be given. If M_1 is a (μ,θ_1)-limit over M and M_2 is a (μ,θ_2)-limit over M, then there exists $g : M_1 \cong M_2$ with $g \restriction M = id_M$.*

A main result of this paper, Corollary 8.1, is a solution to this conjecture under Assumptions 1.2.(1)-(5) plus $\mathcal{K}^{am}_\mu$ is closed under unions of length $< \mu^+$.

We will need one more notion of limit model, which will appear implicitly in the proofs of Theorem 6.7 and Theorem 7.8. This notion is a mild extension of the notion of limit models already defined:

Definition 2.31. Let μ be a cardinal $< \lambda$, we say that $\check{M}$ is a (μ,μ^+) limit over M iff there exists a $\prec_{\mathcal{K}}$-increasing and continuous chain of models $\langle M_i \in \mathcal{K}^{am}_\mu \mid i < \mu^+\rangle$ satisfying

(1) $M_0 = M$
(2) $\bigcup_{i<\mu^+} M_i = \check{M}$ and
(3) for $i < \mu^+$, M_{i+1} is universal over M_i

Remark 2.32. While it is known that (μ,θ)-limit models are amalgamation bases when $\theta < \mu^+$, it is open as to whether or not (μ,μ^+)-limits are amalgamation bases. To avoid confusion between these two concepts of limit models, we will always denote (μ,μ^+)-limit models with a ˇ above the model's name (ie. $\check{M}$)

The existence of (μ,μ^+)-limit models follows from the fact that (μ,θ)-limit models are amalgamation bases when $\theta < \mu^+$, see Corollary 4.9. The uniqueness of (μ,μ^+)-limit models (Proposition 2.33) can be shown using an easy back and forth construction as in the proof of Proposition 2.28.

Proposition 2.33. *Suppose $\check{M}_1$ and $\check{M}_2$ are (μ,μ^+)-limits over M_1 and M_2, respectively. If there exists an isomorphism $h : M_1 \cong M_2$, then h can be extended to an isomorphism $g : \check{M}_1 \cong \check{M}_2$.*

(μ,μ^+)-limit models turn to be useful because they are weakly model homogeneous (in Section 9 we will make this definition more explicit) providing a replacement for monster models:

Proposition 2.34. *If $\check{M}$ is a (μ,μ^+)-limit, then for every $N \prec_{\mathcal{K}} \check{M}$ with $N \in \mathcal{K}^{am}_\mu$, we have that $\check{M}$ is universal over N. Moreover, $\check{M}$ is a (μ,μ^+)-limit over N.*

3. Ehrenfeucht-Mostowski Models

Since $\mathcal{K}$ has no maximal models, $\mathcal{K}$ has models of cardinality $Hanf(\mathcal{K})$. Then by Theorem 3.1, we can construct Ehrenfeucht-Mostowski models.

Theorem 3.1 (Claim 0.6 of [Sh 394] or see [Gr])**.** *Assume that $\mathcal{K}$ is an AEC that contains a model of cardinality $\geq \beth_{(2^{LS(\mathcal{K})})^+}$. Then, there is a Φ, proper for linear orders, such that for linear orders $I \subseteq J$ we have that*

1. $EM(I,\Phi) \restriction L(\mathcal{K}) \prec_{\mathcal{K}} EM(J,\Phi) \restriction L(\mathcal{K})$ *and*
2. $\|EM(I,\Phi) \restriction L(\mathcal{K})\| = |I| + LS(\mathcal{K})$.

We describe an index set which appears often in work toward the categoricity conjecture. This index set was used in [KoSh], [Sh 394] and [ShVi].

Notation 3.2. Let $\alpha < \lambda$ be given. We define

$$I_\alpha := \left\{\ \eta \in {}^\omega\alpha : \{n < \omega \mid \eta[n] \neq 0\} \text{ is finite }\ \right\}$$

Associate with I_α the lexicographical ordering $\lessdot$. If $X \subseteq \alpha$, we write $I_X := \left\{\eta \in {}^\omega\mathrm{X} : \{n < \omega \mid \eta[n] \neq 0\} \text{ is finite}\right\}$.

The following proposition is proved in several papers e.g. [ShVi].

Proposition 3.3. *If $M \prec_{\mathcal{K}} EM(I_\lambda,\Phi) \restriction L(\mathcal{K})$ is a model of cardinality μ^+ with $\mu^+ < \lambda$, then there exists a $\prec_{\mathcal{K}}$-mapping $f : M \to EM(I_{\mu^+},\Phi) \restriction L(\mathcal{K})$.*

A variant of this universality property is (implicit in Lemma 3.7 of [KoSh]):

Proposition 3.4. *Suppose κ is a regular cardinal. If $M \prec_{\mathcal{K}} EM(I_\kappa,\Phi) \restriction L(\mathcal{K})$ is a model of cardinality $< \kappa$ and $N \prec_{\mathcal{K}} EM(I_\lambda,\Phi) \restriction L(\mathcal{K})$ is an extension of M of cardinality $\|M\|$, then there exists a $\prec_{\mathcal{K}}$-embedding $f : N \to EM(I_\kappa,\Phi) \restriction L(\mathcal{K})$ such that $f \restriction M = id_M$.*

4. Amalgamation Bases

Since the amalgamation property for abstract elementary classes is inherent in the definition of types, most work towards understanding AECs has been under the assumption that the class $\mathcal{K}$ has the amalgamation property. In [ShVi], Shelah and Villaveces begin to tackle the categoricity problem with an approach that does not require the amalgamation property as an assumption. Shelah and Villaveces, however, prove a weak amalgamation property, which they refer to as *density of amalgamation bases*, summarized here:

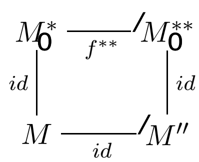
Theorem 4.1 (Theorem 1.2.4 from [ShVi]). *For every $M \in \mathcal{K}_{<\lambda}$, there exists $N \in \mathcal{K}^{am}_{\|M\|}$ with $M \prec_{\mathcal{K}} N$.*

We can now improve Lemma 2.21 slightly. This improvement is used throughout this paper.

Lemma 4.2. *For every μ with $LS(\mathcal{K}) < \mu < \lambda$, if $M \in \mathcal{K}^{am}_{\mu}$, $N \in \mathcal{K}$ and $\bar{a} \in {}^{\mu^+>}\mathrm{N}$ are such that $M \prec_{\mathcal{K}} N$, then there exists $M^{\bar{a}} \in \mathcal{K}^{am}_{\mu}$ such that $M^{\bar{a}}$ is universal over M and $M \bigcup \bar{a} \subseteq M^{\bar{a}}$.*

Proof. By Axiom 6 of AEC, we can find $M' \prec_{\mathcal{K}} N_\lambda$ of cardinality μ containing $M \bigcup \bar{a}$. Applying Theorem 4.1, there exists an amalgamation base of cardinality μ, say M'', extending M'. By Lemma 2.21 we can find a universal extension of M'' of cardinality μ, say $M^{\bar{a}}$.

Notice that $M^{\bar{a}}$ is also universal over M. Why? Suppose M^* is an extension of M of cardinality μ. Since M is an amalgamation base we can amalgamate M'' and M^* over M. WLOG we may assume that the amalgam, M^{**}, is an extension of M'' of cardinality μ and $f^* : M^* \to M^{**}$ with $f^* \restriction M = id_M$.



Now, since $M^{\bar{a}}$ is universal over M'', there exists a $\prec_{\mathcal{K}}$-mapping g such that $g : M^{**} \to M^{\bar{a}}$ with $g \restriction M'' = id_{M''}$. Notice that $g \circ f^*$ gives us the desired mapping of M^* into $M^{\bar{a}}$. ⊣

While Theorem 4.1 asserts the existence of amalgamation bases, it is unknown (in this context) what characterizes amalgamation bases. Shelah and Villaveces have claimed that every limit model is an amalgamation base (Fact 1.3.10 of [ShVi]), under the hypothesis of $\diamondsuit_{S^{\mu^+}_{\mathrm{cf}(\mu)}}$ (which does not follow from GCH). We believe that Assumption 1.2.(5) is needed. We provide a proof that every limit model is an amalgamation base under this additional assumption.

Definition 4.3. Let θ be a regular ordinal $< \mu^+$. We denote

$$S^{\mu^+}_{\theta} := \{\alpha < \mu^+ \mid \mathrm{cf}(\alpha) = \theta\}.$$

Definition 4.4. For μ a cardinal and $S \subseteq \mu^+$ a stationary set, $\Phi_{\mu^+}(S)$ is said to hold iff for all $F : {}^{\lambda^+>}2 \to 2$ there exists $g : \lambda^+ \to 2$ so that for every $f : \lambda^+ \to 2$ the set

$$\{\delta \in S \mid F(f \restriction \delta) = g(\delta)\} \text{ is stationary.}$$

We will be using a consequence of $\Phi_{\mu^+}(S)$, called $\Theta_{\mu^+}(S)$ (see [Gr]).

Definition 4.5. For μ a cardinal $S \subseteq \mu^+$ a stationary set, $\Theta_{\mu^+}(S)$ is said to hold if and only if for all families of functions

$$\{f_\eta \ : \ \eta \in {}^{\mu^+}2 \text{ where } f_\eta : \mu^+ \to \mu^+\}$$

and for every club $C \subseteq \mu^+$, there exist $\eta \neq \nu \in {}^{\mu^+}2$ and there exists a $\delta \in C \cap S$ such that

(1) $\eta \restriction \delta = \nu \restriction \delta$,
(2) $f_\eta \restriction \delta = f_\nu \restriction \delta$ and
(3) $\eta[\delta] \neq \nu[\delta]$.

Fact 4.6. $\diamondsuit_{\mu^+}(S^{\mu^+}_\theta) \Longrightarrow \Phi_{\mu^+}(S^{\mu^+}_\theta) \Longrightarrow \Theta_{\mu^+}(S^{\mu^+}_\theta)$.

Theorem 4.7. *If M is a (μ,θ)-limit for some θ with $\theta < \mu^+ \leq \lambda$, then M is an amalgamation base.*

This Theorem appears in [ShVi] with a one-line proof. J. Baldwin has requested that I supply the complete proof here. Before we begin the proof notice that:

Remark 4.8 (Invariance)**.** By Axiom 1 of AEC, if M is an amalgamation base and f is an $\prec_{\mathcal{K}}$-embedding, then $f(M)$ is an amalgamation base.

Proof of Theorem 4.7. Given μ, suppose that θ is the minimal infinite ordinal $< \mu^+$ such that there exists a model M which is a (μ,θ)-limit and not an amalgamation base. Notice that by Proposition 2.29, we may assume that $\mathrm{cf}(\theta) = \theta$.

Now we define by induction on the length of $\eta \in {}^{\mu^+>}2$ a tree of structures, $\langle M_\eta \mid \eta \in {}^{\mu^+>}2\rangle$, satisfying:

(1) for $\eta \lessdot \nu \in {}^{\mu^+>}2$, $M_\eta \prec_{\mathcal{K}} M_\nu$
(2) for $l(\eta)$ a limit ordinal with $\mathrm{cf}(l(\eta)) \leq \theta$, $M_\eta = \bigcup_{\alpha<l(\eta)} M_{\eta\restriction\alpha}$
(3) for $\eta \in {}^{\alpha}2$ with $\alpha \in S^{\mu^+}_\theta$,
  (a) M_η is a (μ,θ)-limit model
  (b) $M_{\eta\hat{}0}, M_{\eta\hat{}1}$ cannot be amalgamated over M_η
  (c) $M_{\eta\hat{}0}$ and $M_{\eta\hat{}1}$ are amalgamation bases of cardinality μ
(4) for $\eta \in {}^{\alpha}2$ with $\alpha \notin S^{\mu^+}_\theta$,
  (a) M_η is an amalgamation base
  (b) $M_{\eta\hat{}0}, M_{\eta\hat{}1}$ are universal over M_η and
  (c) $M_{\eta\hat{}0}$ and $M_{\eta\hat{}1}$ are amalgamation bases of cardinality μ (it may be that $M_{\eta\hat{}0} = M_{\eta\hat{}1}$ in this case).

This construction is possible:

$\eta = \langle\rangle$: By Theorem 4.1, we can find $M' \in \mathcal{K}^{am}_\mu$ such that $M \prec_{\mathcal{K}} M'$. Define $M_{\langle\rangle} := M'$.

$l(\eta)$ is a limit ordinal: When $\mathrm{cf}(l(\eta)) > \theta$, let $M'_\eta := \bigcup_{\alpha<l(\eta)} M_{\eta\restriction\alpha}$. M'_η is not necessarily an amalgamation base, but for the purposes of this construction, continuity at such limits is not important. Thus we can find an

extension of M'_η, say M_η, of cardinality μ where M_η is an amalgamation base.

For η with $\mathrm{cf}(l(\eta)) \leq \theta$, we require continuity. Define $M_\eta := \bigcup_{\alpha<l(\eta)} M_{\eta\restriction\alpha}$. We need to verify that if $l(\eta) \notin S^{\mu^+}_\theta$, then M_η is an amalgamation base. In fact, we will show that such a M_η will be a $(\mu, \mathrm{cf}(l(\eta)))$-limit model. Let $\langle \alpha_i \mid i < \mathrm{cf}(l(\eta))\rangle$ be an increasing and continuous sequence of ordinals converging to $l(\eta)$ such that $\mathrm{cf}(\alpha_i) < \theta$ for every $i < \mathrm{cf}(l(\eta))$. Condition (4b) guarantees that for $i < \mathrm{cf}(l(\eta))$, $M_{\eta\restriction\alpha_{i+1}}$ is universal over $M_{\eta\restriction\alpha}$. Additionally, condition (2) ensures us that $\langle M_{\eta\restriction\alpha_i} \mid i < \mathrm{cf}(l(\eta))\rangle$ is continuous. This sequence of models witnesses that M_η is a $(\mu, \mathrm{cf}(l(\eta)))$-limit model. By our minimal choice of θ, we have that $(\mu, \mathrm{cf}(l(\eta)))$-limit models are amalgamation bases.

$\eta\hat{\ }i$ where $l(\eta) \in S^{\mu^+}_\theta$: We first notice that $M_\eta := \bigcup_{\alpha<l(\eta)} M_{\eta\restriction\alpha}$ is a (μ, θ)-limit model. Why? Since $l(\eta) \in S^{\mu^+}_\theta$ and θ is regular, we can find an increasing and continuous sequence of ordinals, $\langle \alpha_i \mid i < \theta\rangle$ converging to $l(\eta)$ such that for each $i < \theta$ we have that $\mathrm{cf}(\alpha_i) < \theta$. Condition (4b) of the construction guarantees that for each $i < \theta$, $M_{\eta\restriction\alpha_{i+1}}$ is universal over $M_{\eta\restriction\alpha_i}$. Thus $\langle M_{\eta\restriction\alpha_i} \mid i < \theta\rangle$ witnesses that M_η is a (μ, θ)-limit model.

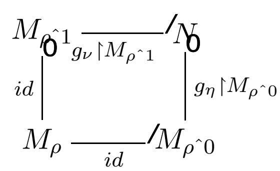
Since M_η is a (μ, θ)-limit, we can fix an isomorphism $f : M \cong M_\eta$. By Remark 4.8, M_η is not an amalgamation base. Thus there exist $M_{\eta\hat{\ }0}$ and $M_{\eta\hat{\ }1}$ extensions of M_η which cannot be amalgamated over M_η. WLOG we can choose, $M_{\eta\hat{\ }0}$ and $M_{\eta\hat{\ }1}$ to be elements of $\mathcal{K}^{am}_\mu$.

$\eta\hat{\ }i$ where $l(\eta) \notin S^{\mu^+}_\theta$: Since M_η is an amalgamation base, we can choose $M_{\eta\hat{\ }0}$ and $M_{\eta\hat{\ }1}$ to be extensions of M_η such that $M_{\eta\hat{\ }l} \in \mathcal{K}^{am}_\mu$ and $M_{\eta\hat{\ }l}$ is universal over M_η, for $l = 0, 1$.

This completes the construction. For every $\eta \in {}^{\mu^+}2$, define $M_\eta := \bigcup_{\alpha<\mu^+} M_{\eta\restriction\alpha}$. By categoricity in λ and Proposition 3.3, we can fix a $\prec_{\mathcal{K}}$-mapping $g_\eta : M_\eta \to EM(I_{\mu^+}, \Phi) \restriction L(\mathcal{K})$ for each $\eta \in {}^{\mu^+}2$. Now apply $\Theta_{\mu^+}(S^{\mu^+}_\theta)$ to find $\eta, \nu \in {}^{\mu^+}2$ and $\alpha \in S^{\mu^+}_\theta$ such that

- $\rho := \eta \restriction \alpha = \nu \restriction \alpha$,
- $\eta[\alpha] = 0$, $\nu[\alpha] = 1$ and
- $g_\eta \restriction M_\rho = g_\nu \restriction M_\rho$.

By Axiom 6 (the Löwenheim-Skolem property) of AEC, there exists $N \prec_{\mathcal{K}} EM(I_{\mu^+}, \Phi) \restriction L(\mathcal{K})$ of cardinality μ such that the following diagram commutes:



Notice that $g_\eta \restriction M_{\rho\hat{}0}$, $g_\nu \restriction M_{\rho\hat{}1}$ and N witness that $M_{\rho\hat{}0}$ and $M_{\rho\hat{}1}$ can be amalgamated over M_ρ. Since $l(\rho) = \alpha \in S^{\mu^+}_\theta$, we contradict condition (3b) of the construction.

$\dashv$

Corollary 4.9 (Existence of limit models and (μ, μ^+)-limit models)**.** *For every cardinal μ and limit ordinal θ with $\theta \leq \mu^+ \leq \lambda$, if M is an amalgamation base of cardinality μ, then there exists $M' \in \mathcal{K}$ which is a (μ, θ)-limit over M.*

Proof. By repeated applications of Lemma 2.21 and Theorem 4.7. $\dashv$

5. Weak Disjoint Amalgamation

Shelah and Villaveces prove a version of weak disjoint amalgamation in an attempt to prove an extension property for towers. We will be using weak disjoint amalgamation and provide a proof here for completeness.

Theorem 5.1 (Weak Disjoint Amalgamation [ShVi])**.** *Given $\lambda > \mu \geq LS(\mathcal{K})$ and $\alpha, \theta_0 < \mu^+$ with θ_0 regular. If M_0 is a (μ, θ_0)-limit and $M_1, M_2 \in \mathcal{K}_\mu$ are $\prec_\mathcal{K}$-extensions of M_0, then for every $\bar{b} \in {}^\alpha(M_1 \backslash M_0)$, there exist M_3, a model, and h, a $\prec_\mathcal{K}$-embedding, such that*

1. *$h : M_2 \to M_3$;*
2. *$h \restriction M_0 = id_{M_0}$ and*
3. *$h(M_2) \cap \bar{b} = \emptyset$ (equivalently $h(M_2) \cap M_1 = \emptyset$).*

Shelah and Villaveces provide a proof of this theorem in [ShVi]. It has been suggested that I elaborate on the proof here:

Proof. Suppose that M_0, M_1, M_2 and $\bar{b} \in M_1$ form a counter-example. Since M_0 is a μ amalgamation base, we may assume that there exists $M^* \in \mathcal{K}_\mu$ with $M_1, M_2 \prec_\mathcal{K} M^*$. Let θ be regular and $< \mu^+$ such that M_0 is a (μ, θ)-limit. We define a $\prec_\mathcal{K}$-increasing and continuous sequence of models $\langle N_i \mid i < \mu^+\rangle$ satisfying:

1. $N_i \in \mathcal{K}^{am}_\mu$
2. N_{i+1} is universal over N_i and
3. when $\mathrm{cf}(i) = \theta$, we additionally define N^1_i, N^2_i, N^*_i and $\bar{b}_i \in N^1_i$ such that there exists an isomorphism $f : M^* \cong N^*_i$ with $f(M_0) = N_i$, $f(M_1) = N^1_i$, $f(M_2) = N^2_i$ and $f(\bar{b}) = \bar{b}_i$.

The construction is possible by Lemma 2.21, Theorem 4.7 and Proposition 2.28.

Let $N_{\mu^+} := \bigcup_{i<\mu^+} N_i$. Since $\mathcal{K}$ is categorical in λ, Proposition 3.3 allows us to find a $\prec_\mathcal{K}$ $g : N_{\mu^+} \to EM(I^+_\mu, \Phi) \restriction L(\mathcal{K})$. So WLOG, we may assume that $N_{\mu^+} \prec_\mathcal{K} EM(I^+_\mu, \Phi) \restriction L(\mathcal{K})$.

Let $E \subseteq \mu^+$ be a club such that

$$\delta \in E \Rightarrow N_\delta \prec_\mathcal{K} EM(I_\delta, \Phi) \restriction L(\mathcal{K}).$$

For each $i \in S^{\mu^+}_\theta$, choose a Skolem-term τ_i and a sequence of indices $\alpha_{i,0},\dots,\alpha_{i,n_i-1}$ such that $\bar{b} = \tau_i(\alpha_{i,0},\dots,\alpha_{i,n_i-1})$. Let $m_i < n_i$ be such

$$k < m_i \Leftrightarrow \alpha_{i,k} \in I_i.$$

Set $\alpha_{i,<m_i} := \langle \alpha_{i,k} \mid 0 \leq k < m_i\rangle$ and $\alpha_{i,\geq m_i} := \langle \alpha_{i,k} \mid m_i \leq k < n_i\rangle$.

Let $\delta_0 \in E \cap S^{\mu^+}_\theta$.

For every δ_1, with $\delta_0 < \delta_1 < \mu^+$. Define g_{δ_1} to be the $\prec_{\mathcal{K}}$-mapping from $EM(I_{\delta_1},\Phi)\restriction L(\mathcal{K})$ to $EM(I_{\mu^+},\Phi)\restriction L(\mathcal{K})$ induced by the mapping from μ^+ to μ^+ defined by

$$j \mapsto \begin{cases} j & \text{if } j < \delta_0 \\ \delta_1 + j & \text{if } \delta_0 \leq j < \delta_1 \end{cases}$$

Subclaim 5.2. *There exists some $\delta_1 < \mu^+$ such that $g_{\delta_1}(N^1_{\delta_0}) \cap \bar{b}_{\delta_0} = \emptyset$.*

Proof. Suppose the claim fails. Then for every δ with $\delta_0 < \delta < \mu^+$, there exists a Skolem term σ_δ and a sequence of indices

$$\beta_{\delta,0},\dots,\beta_{\delta,m_\delta-1},\beta_{\delta,m_\delta},\dots,\beta_{\delta,n_\delta-1}$$

such that

$$k < m_\delta \Leftrightarrow \beta_{\delta,k} \in I_{\delta_0}$$

and $b = \sigma_{\delta_0}(\beta_{\delta,0},\dots,\beta_{\delta,n_\delta-1})$.

Let $\beta_{\delta,<m_\delta} := \langle \beta_{\delta,k} \mid 0 \leq k < m_\delta\rangle$ and $\beta_{\delta,\geq m_\delta} := \langle \beta_{\delta,k} \mid m_\delta \leq k < n_\delta\rangle$.

Notice that

$$EM(I_{\mu^+},\Phi)\restriction L(\mathcal{K}) \models b = \sigma_{\delta_0}(\beta_{\delta,<m_\delta};\beta_{\delta,\geq m_\delta}) = \tau_{\delta_0}(\alpha_{\delta_0,<m_{\delta_0}};\alpha_{\delta_0,\geq m_{\delta_0}}).$$

By our definition of g_δ, we have that

$$(*)_\delta \quad k \geq m_\delta \Leftrightarrow \beta_{\delta,k} \in I_{\delta\setminus\delta_1\cup\delta_0}.$$

In other words when $k \geq m_\delta$, every term from the sequence $\beta_{\delta,k}$ which is larger than δ_0 is also larger than δ_1. Thus, for $k \geq m_\delta$, $\beta_{\delta,k}$ and $\alpha_{\delta_0\geq m_{\delta_0}}$ are not intertwined above δ_0.

Since all our indices are finite sequences and δ_0 is a limit ordinal, there exists $\delta^* < \delta_1$ such that $\alpha_{\delta_0,<m_{\delta_0}},\beta_{\delta,<m_\delta} \in I_{\delta^*}$. This allows us to find a sequence of indices $\alpha^* \in I_{\delta_0}$ which have the same type over I_{δ^*} (with respect to the lexicographical ordering) as $\alpha_{\delta_0,\geq m_{\delta_0}}$. So by indiscernibility and $(*)_\delta$

$$EM(I_{\mu^+},\Phi)\restriction L(\mathcal{K}) \models \sigma_{\delta_0}(\beta_{\delta,<m_\delta};\beta_{\delta,\geq m_\delta}) = \tau_{\delta_0}(\alpha_{\delta_0,<m_{\delta_0}};\alpha^*).$$

Notice that $\sigma_{\delta_0}(\beta_{\delta,<m_\delta};\beta_{\delta,\geq m_\delta}) = b$. Thus we have found a way to construct b from I_{δ_0} (by $\tau_{\delta_0}(\alpha_{\delta_0,<m_{\delta_0}};\alpha^*)$). This contradicts our choice of $b \notin EM(I_{\delta_0})\restriction L(\mathcal{K})$.

$\dashv$

Let g' be an order preserving mapping and α_2 an ordinal such that

(1) $\alpha_1 < \alpha_2 < \mu^+$,
(2) $g' : \alpha_1 \to \alpha_2$,
(3) $g' \restriction \alpha_0 = id_{\alpha_0}$ and

(4) $g'(\alpha_1) \cap \alpha_1 = \alpha_0$.

Let g^* be the $\prec_{\mathcal{K}}$-mapping taking M_1^* to $EM(I_{\alpha_2}, \Phi) \restriction L(\mathcal{K})$ determined by g'. Notice that by Subclaim 5.2, g^* witnesses that M_0^*, M_1^*, M_2^* and $\bar{b}^*$ can be weakly disjointly amalgamated.

$\dashv$

Let us state an easy corollary of Theorem 5.1 that will simplify future constructions:

Corollary 5.3. *Suppose μ, M_0, M_1, M_2 and $\bar{b}$ are as in the statement of Theorem 5.1. If $\check{M}$ is universal over M_1, then there exists a $\prec_{\mathcal{K}}$-mapping h such that*

(1) $h : M_2 \to \check{M}$,
(2) $h \restriction M_0 = id_{M_0}$ *and*
(3) $h(M_2) \cap \bar{b} = \emptyset$ *(equivalently $h(M_2) \cap M_1 = \emptyset$).*

Proof. By Theorem 5.1, there exists a $\prec_{\mathcal{K}}$-mapping g and a model M_3 of cardinality μ such that

· $g : M_2 \to M_3$
· $g \restriction M_0 = id_{M_0}$
· $g(M_2) \cap \bar{b} = \emptyset$ and
· $M_1 \prec_{\mathcal{K}} M_3$.

Since $\check{M}$ is universal over M_1, we can fix a $\prec_{\mathcal{K}}$-mapping f such that

· $f : M_3 \to \check{M}$ and
· $f \restriction M_1 = id_{M_1}$

Notice that $h := g \circ f$ is the desired mapping from M_2 into $\check{M}$.

$\dashv$

6. $<^b_{\mu,\alpha}$-EXTENSION PROPERTY FOR $\mathcal{K}^*_{\mu,\alpha}$

Shelah introduced towers in [Sh 48] and [Sh 87b] as a tool to build a model of cardinality μ^+ from models of cardinality μ. Roughly speaking, we will use an increasing and continuous chain (of length σ_1) of towers which have length σ_2 to construct an array of models of height σ_1 and width σ_2 in such a way that the union will simultaneously be a (μ, σ_1)-limit model and a (μ, σ_2)-limit model. The construction of such a model is sufficient to prove the uniqueness of limit models by Lemma 2.28. First we fix the following notation:

Definition 6.1 (Towers Definition 3.1.1 of [ShVi])**.** Let $\mu > LS(\mathcal{K})$ and $\alpha, \theta < \mu^+$

(1)

$$\mathcal{K}_{\mu,\alpha} := \left\{ (\bar{M},\bar{a}) \;\middle|\; \begin{array}{l} (\bar{M},\bar{a}) := (\langle M_\gamma \mid \gamma<\alpha\rangle, \langle a_\gamma \mid \gamma<\alpha\rangle); \\ \bar{M} \text{ is } \prec_{\mathcal{K}} -\text{increasing}; \\ \text{for every } \gamma<\alpha,\ a_\gamma \in M_{\gamma+1}\backslash M_\gamma; \\ \text{for every } \gamma<\alpha,\ M_\gamma \in \mathcal{K}_\mu \end{array} \right\}$$

(2) $\mathcal{K}^\theta_{\mu,\alpha} := \{(\bar{M},\bar{a}) \in \mathcal{K}_{\mu,\alpha} \mid \text{for every } \gamma<\alpha,\ M_\gamma \text{ is a } (\mu,\theta)\text{-limit}\}$

(3) $\mathcal{K}^*_{\mu,\alpha} := \bigcup_{\theta<\mu^+} \mathcal{K}^\theta_{\mu,\alpha}$

Fact 6.2 (Fact 3.17 from [ShVi]). *Suppose $\mathcal{K}$ is categorical in λ. Given $\lambda > \mu \geq LS(\mathcal{K})$, $\alpha < \mu^+$ and θ a regular cardinal with $\theta < \mu^+$, we have that $\mathcal{K}^\theta_{\mu,\alpha} \neq \emptyset$.*

In order to have some control during the construction of the array of models, an ordering on the towers is defined. Namely, we would like to have continuity at limit stages of the chain. So we would like that the union of the increasing chain of towers, should be a tower. The problem is that the models at the limit stages may not be limit models. This motivates the following ordering on towers:

Definition 6.3 (Definition 3.1.3 of [ShVi]). For $(\bar{M},\bar{a}),(\bar{N},\bar{b}) \in \mathcal{K}^*_{\mu,\alpha}$ we say that

(1) $(\bar{M},\bar{a}) \leq^b_{\mu,\alpha} (\bar{N},\bar{b})$ if and only if
   (a) $\bar{a}=\bar{b}$;
   (b) for every $\gamma<\alpha$, $M_\gamma \preceq_{\mathcal{K}} N_\gamma$ and
   (c) whenever $M_\gamma \prec_{\mathcal{K}} N_\gamma$, then N_γ is universal over M_γ.

(2) $(\bar{M},\bar{a}) <^b_{\mu,\alpha} (\bar{N},\bar{b})$ if and only if $(\bar{M},\bar{a}) \leq^b_{\mu,\alpha} (\bar{N},\bar{b})$ and for every $\gamma<\alpha$, $M_\gamma \neq N_\gamma$.

Remark 6.4. If $\langle(\bar{M},\bar{a})_\sigma \in \mathcal{K}^*_{\mu,\alpha} \mid \sigma<\gamma\rangle$ is a $<^b_{\mu,\alpha}$-increasing and continuous chain with $\gamma<\mu^+$, then $\bigcup_{\sigma<\gamma}(\bar{M},\bar{a})_\sigma \in \mathcal{K}^*_{\mu,\alpha}$. Why? Notice that for $i<\alpha$, $M_{i,\gamma} := \bigcup_{\sigma<\gamma} M_{i,\sigma}$ is a limit model, witnessed by $\langle M_{i,\sigma} \mid \sigma<\gamma\rangle$.

In order to construct a non-trivial chain of towers, we need to be able to take proper $<^b_{\mu,\alpha}$-extensions.

Definition 6.5. We say the $<^b_{\mu,\alpha}$-*extension property* holds iff for every $(\bar{M},\bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ there exists $(\bar{M}',\bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ such that $(\bar{M},\bar{a}) <^b_{\mu,\alpha} (\bar{M}',\bar{a})$.

Remark 6.6. Shelah and Villaveces claim the $<^b_{\mu,\alpha}$-extension property as Fact 3.19(1) in [ShVi]. Their proof does not converge. As of the Fall of 2001, they were unable to produce a proof of this claim.

We will prove the $<^b_{\mu,\alpha}$-extension property for a particular class of towers:

Theorem 6.7 (The $<^b_{\mu,\alpha}$-extension property for nice towers). *For every nice $(\bar{M},\bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$, there exists a nice tower $(\bar{M}',\bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ such that $(\bar{M},\bar{a}) <^b_{\mu,\alpha} (\bar{M}',\bar{a})$*

Definition 6.8. $(\langle M_i \mid i < \alpha\rangle, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ is *nice* provided that for every limit ordinal $i < \alpha$, we have that $\bigcup_{j<i} M_j$ is an amalgamation base.

Notice that in the definition of towers, we do not require continuity at limit ordinals i of the sequence of models. This allows for towers in which $M_i \neq \bigcup_{j<i} M_j$. Since we only require that M_i is an amalgamation base, there are towers which are not necessarily nice.

Remark 6.9. Why isn't Theorem 6.7 sufficient? What is needed to conclude the uniqueness of limit models?

1. (1) We would like to ultimately construct a $<^b_{\mu,\alpha}$-increasing and continuous chain of towers. The problem occurs at limit stages. Suppose we have constructed a $<^b_{\mu,\alpha}$-increasing and continuous sequence of towers, $\langle(\bar{M}^n, \bar{a}) \mid n \leq \omega\rangle$. We may not be able to extend this chain of towers because we are not guaranteed that the limit $(\bar{M}^\omega, \bar{a})$ is a nice tower. A sufficient resolution would be to show that every tower has a $\leq^b_{\mu,\alpha}$-extension which is nice.
2. (2) Alternatively, if we try to carry out the arguments of [ShVi] towards the uniqueness of limit models using only nice towers, we see that the only obstacle is to prove that reduced towers are continuous.
3. (3) If we make an additional assumption, that $\mathcal{K}^{am}_\mu$ is closed under unions of length $< \mu^+$, then we can adjust the proof of Theorem 6.7 and eliminate the assumption of niceness, thereby getting the full $<^b_{\mu,\alpha}$-extension property.

Proof of Theorem 6.7. Let μ be a cardinal and α a limit ordinal such that $\alpha < \mu^+ \leq \lambda$. Let a nice tower $(\bar{M}, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ be given. Denote by M_α a model in $\mathcal{K}^{am}_\mu$ extending $\bigcup_{i<\alpha} M_i$. Fix $\check{M}$ to be a (μ, μ^+)-limit model over M_α.

We define by induction on $i < \alpha$ a sequence of models $\langle M'_i \mid i < \alpha\rangle$ and a sequences of $\prec_{\mathcal{K}}$-mappings, $\langle f'_{j,i} \mid j < i < \alpha\rangle$ and $\langle \check{f}_{j,i} \mid j < i < \alpha\rangle$ such that for $i \leq \alpha$:

1. (1) $(\langle f'_{j,i}(M'_j) \mid j \leq i\rangle, \bar{a} \restriction i)$ is a $<^b_{\mu,i}$-extension of $(\bar{M}, \bar{a}) \restriction i$,
2. (2) $(\langle M'_j \mid j < i\rangle, \langle f'_{j,i} \mid j \leq i\rangle)$ forms a directed system,
3. (3) M'_i is universal over M_i,
4. (4) M'_{i+1} is universal over $f'_{i,i+1}(M'_i)$,
5. (5) $f'_{j,i} \restriction M_j = id_{M_j}$,
6. (6) $M'_i \prec_{\mathcal{K}} \check{M}$,
7. (7) $f_{j,i}$ can be extended to an automorphism of $\check{M}$, $\check{f}_{j,i}$, for $j \leq i$ and
8. (8) $(\langle \check{M}_j = \check{M} \mid j < i\rangle, \langle \check{f}_{k,j} \mid k \leq j < i\rangle)$ forms a directed system.

Notice that the M'_i's will not necessarily form an extension of the tower $(\bar{M}, \bar{a})$. Rather, for each $i < \alpha$, we find some image of $\langle M_j \mid j < i\rangle$ which will extend the initial segment of length i of $(\bar{M}, \bar{a})$ (see condition (1) of the construction).

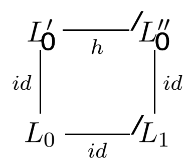
The construction is possible:

$i = 0$: Since M_0 is an amalgamation base, we can find $M_0'' \in \mathcal{K}^*_\mu$ (a first approximation of the desired M_0') such that M_0'' is universal over M_0. By Corollary 5.3 (applied to M_0, M_α, M_0'' and $\bar{a}$), we can find a $\prec_{\mathcal{K}}$-mapping $h : M_0'' \to \check{M}$ such that $h \restriction M_0 = id_{M_0}$ and $h(M_0'') \cap \bar{a} = \emptyset$. Set $M_0' := h(M_0'')$, $f'_{0,0} := id_{M_0'}$ and $\check{f}_{0,0} := id_{\check{M}}$.

$i = j + 1$: Suppose that we have completed the construction of all $k \leq j$. Since M_j' and M_{j+1} are both $\prec_{\mathcal{K}}$-substructures of $\check{M}$, we can get M''_{j+1} (a first approximation to the desired M'_{j+1}) such that $M''_{j+1} \in \mathcal{K}^*_\mu$ is universal over M_j' and universal over M_{j+1}. How? By the Downward Löwenheim Skolem Axiom (Axiom 6) of AEC and the density of amalgamation bases (Theorem 4.1), we can find an amalgamation base L of cardinality μ such that $M_j', M_{j+1} \prec_{\mathcal{K}} L$. By Lemma 2.21 and Corollary 4.9, there exists M''_{j+1}, a (μ, ω)-limit over L.

Subclaim 6.10. *M''_{j+1} is universal over M_j' and is universal over M_{j+1}.*

Proof. It suffices to show that when $L_0 \prec_{\mathcal{K}} L_1 \prec_{\mathcal{K}} L$ are amalgamation bases of cardinality μ, if L is universal over L_1, then L is universal over L_0. Let L' be an extension of L_0 of cardinality μ. Since L_0 is an amalgamation base, we can find an amalgam L'' such that the following diagram commmutes:



Since L is universal over L_1, there exists $g : L'' \to L$ with $g \restriction L_1 = id_{L_1}$. Notice that $g \circ h : L' \to L$ with $g \circ h \restriction L_0 = id_{L_0}$. ⊣

M''_{j+1} may serve us well if it does not contain any a_l for $j + 1 \leq l < \alpha$, but this is not guaranteed. So we need to make an adjustment. Notice that $\check{M}$ is universal over M_{j+1}. Thus we can apply Corollary 5.3 to M_{j+1}, M_α, M''_{j+1} and $\langle a_l \mid j + 1 \leq l < \alpha \rangle$. This yields a $\prec_{\mathcal{K}}$-mapping h such that

- $h : M''_{j+1} \to \check{M}$
- $h \restriction M_{j+1} = id_{M_{j+1}}$ and
- $h(M''_{j+1}) \cap \{a_l \mid j + 1 \leq l < \alpha\} = \emptyset$.

Set $M'_{j+1} := h(M''_{j+1})$, $f'_{j+1,j+1} = id_{M_{j+1}}$, $\check{f}_{j+1,j+1} = id_{\check{M}}$ and $f'_{j,j+1} := h \restriction M_j'$. Since $\check{M}$ is a (μ, μ^+)-limit over both M_j' and $f'_{j,j+1}(M_j')$, by Proposition 2.33 we can extend $f'_{j,j+1}$ to an automorphism of $\check{M}$, denoted by $\check{f}_{j,j+1}$.

To guarantee that we have a directed system, for $k < j$, define $f'_{k,j+1} := f'_{j,j+1} \circ f'_{k,j}$ and $\check{f}_{k,j+1} := \check{f}_{j,j+1} \circ \check{f}_{k,j}$.

i is a limit ordinal: Suppose that $(\langle M_j' \mid j < i \rangle, \langle f'_{k,j} \mid k \leq j < i \rangle)$ and $(\langle \check{M} \mid j < i \rangle, \langle \check{f}_{k,j} \mid k \leq j < i \rangle)$ have been defined. Since they are both directed systems, we can take direct limits.

Claim 6.11. *We can choose the direct limits* $(M^*_i, \langle f^*_{j,i} \mid j \leq i\rangle)$ *and* $(\check{M}^*_i, \langle \check{f}^*_{j,i} \mid j \leq i\rangle)$ *of* $(\langle M'_j \mid j < i\rangle, \langle f'_{k,j} \mid k \leq j < i\rangle)$ *and* $(\langle \check{M} \mid j < i\rangle, \langle \check{f}_{k,j} \mid k \leq j < i\rangle)$ *respectively such that*

- $M^*_i \prec_{\mathcal{K}} \check{M}^*_i$
- $f^*_{j,i}$ *is an automorphism for every* $j \leq i$
- $\check{M}^*_i = \check{M}$
- $f^*_{j,i} \restriction M_j = id_{M_j}$ *for every* $j < i$.

Proof. We will realize each of these conditions one at a time. By Lemma 2.7 we may choose direct limits $(M^{**}_i, \langle f^{**}_{j,i} \mid j \leq i\rangle)$ and $(\check{M}^{**}_i, \langle \check{f}^{**}_{j,i} \mid j \leq i\rangle)$ such that $M^{**}_i \prec_{\mathcal{K}} \check{M}^{**}_i$. By Claim 2.8 we have that for every $j \leq i$, $\check{f}^{**}_{j,i}$ is an automorphism and $\check{M}^{**}_i = \check{M}$. Notice that this forms a direct limit satisfying the first three properties.

Subclaim 6.12. $\langle f^{**}_{j,i} \restriction M_j \mid j < i\rangle$ *is increasing.*

Proof. Let $j < k < i$ be given. By construction

$$f'_{j,k} \restriction M_j = id_{M_j}.$$

An application of $f^{**}_{k,i}$ yields

$$f^{**}_{k,i} \circ f'_{j,k} \restriction M_j = f^{**}_{k,i} \restriction M_j.$$

By the definition of directed limits, we have

$$f^{**}_{j,i} \restriction M_j = f^{**}_{k,i} \circ f'_{j,k} \restriction M_j = f^{**}_{k,i} \restriction M_j.$$

⊣

By the subclaim, we have that $g := \bigcup_{j<i} f^{**}_{j,i} \restriction M_j$ is a partial automorphism of $\check{M}$ from $\bigcup_{j<i} M_j$ onto $\bigcup_{j<i} f^{**}_{j,i}(M_j)$. Since $\check{M}$ is a (μ, μ^+)-limit model and since $\bigcup_{j<i} M_j$ is an amalgamation base we can extend g to $G \in Aut(\check{M})$ by Proposition 2.33. Now consider the direct limit defined by $M^*_i := G^{-1}(M^{**}_i)$ with $\langle f^*_{j,i} := G^{-1} \circ f^{**}_{j,i} \mid j < i\rangle$ and $f^*_{i,i} = id_{M^*_i}$ and the direct limit $\check{M}^*_i := \check{M}$ with $\langle \check{f}^*_{j,i} := G^{-1} \circ \check{f}^{**}_{j,i} \mid j < i\rangle$ and $\check{f}^*_{i,i} := id_{N^*_i}$. Notice that $f^*_{j,i} \restriction M_j = G^{-1} \circ f^{**}_{j,i} \restriction M_j = id_{M_j}$ for $j < i$.

⊣

By Condition (4) of the construction, notice that M^*_i is a (μ, i)-limit model witnessed by $\langle f^*_{j,i}(M'_j) \mid j < i\rangle$. Hence M^*_i is an amalgamation base. Since M^*_i and M_i both live inside of $\check{M}$, we can find $M''_i \in \mathcal{K}^*_\mu$ which is universal over M_i and universal over M^*_i.

By Corollary 5.3 applied to M_i, M_α, M''_i and $\langle a_l \mid l \leq i < \alpha\rangle$ we can find $h : M''_i \to \check{M}$ such that $h \restriction M_i = id_{M_i}$ and $h(M''_i) \cap \{a_l \mid i \leq l < \alpha\} = \emptyset$.

Set $M'_i := h(M''_i)$, $f'_{i,i} := id_{M_{i,i}}$, $\check{f}_{i,i} := id_{\check{M}}$ and for $j < i$, $f'_{j,i} := h \circ f^*_{j,i}$. We need to verify that for $j \leq i$, $f'_{j,i}(M'_j) \bigcap \{a_l \mid j \leq l < \alpha\} = \emptyset$. Clearly by our application of weak disjoint amalgamation, we have that for every l with $i \leq l < \alpha$ and every $j \leq i$, $a_l \notin M'_i \supseteq f'_{j,i}(M'_j)$. Suppose that $j < i$ and

l is such that $j \leq l < i$. By construction $a_l \notin f'_{j,l+1}(M'_j)$ and $f'_{l+1,i}(a_l) = a_l$. So $f'_{j,i}(M'_j) = f'_{l+1,i} \circ f'_{j,l+1}(M'_j)$ implies that $a_l \notin f'_{j,i}(M'_j)$.

Notice that for every $j < i$, $\check{M}$ is a (μ, μ^+)-limit over both M'_j and $f'_{j,i}(M'_j)$. Thus by the uniqueness of (μ, μ^+)-limit models, we can extend $f'_{j,i}$ to an automorphism of $\check{M}$, denoted by $\check{f}_{j,i}$.

The construction is enough: Let M'_α and $\langle f_{i,\alpha} \mid i \leq \alpha \rangle$ be the direct limit of $(\langle M'_i \mid i < \alpha \rangle, \langle f_{j,i} \mid j \leq i < \alpha \rangle)$. By Subclaim 6.12 we may assume that $\bigcup_{i<\alpha} M_i \prec_{\mathcal{K}} M'_\alpha$. It is routine to verify that $(\langle f_{i,\alpha}(M'_i) \mid i < \alpha \rangle, \bar{a})$ is a $<^b_{\mu,\alpha}$-extension of $(\bar{M}, \bar{a})$.

$\dashv$

7. $<^c_{\mu,\alpha}$ Extension Property for $\mathcal{K}^*_{\mu,\alpha}$

Unfortunately, it seems that working with the relatively simple $\mathcal{K}^*_{\mu,\alpha}$ towers is not sufficient to carry out the proof for the uniqueness of limit models. Shelah and Villaveces have idenitified a more elaborate tower. The extension proprerty for these towers is also missing from [ShVi]. We provide a partial solution to this extension property, analagous to the solution for $\mathcal{K}^*_{\mu,\alpha}$ in the previous section.

While there is no known notion of forking in the general context of AECs, Shelah has generalized the notion of splitting to AECs in [Sh 394]:

Definition 7.1. Let μ be a cardinal with $\mu < \lambda$. For $M \in \mathcal{K}^{am}$ and $p \in$ ga-S(M), we say that p *μ-splits over N* iff $N \prec_{\mathcal{K}} M$ and there exist $N_1, N_2 \in \mathcal{K}_\mu$ and a $\prec_{\mathcal{K}}$-mapping $h : N_1 \cong N_2$ such that

(1) $h(p \restriction N_1) \neq p \restriction N_2$,
(2) $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$ and
(3) $h \restriction N = id_N$.

This notion behaves nicely under stability assumptions. Shelah and Villaveces draw a connection between categoricity and superstability-like properties with:

Theorem 7.2 (Theorem 2.2.1 from [ShVi])**.** *Assume that*

(1) $\langle M_i \mid i \leq \sigma \rangle$ *is $\prec_{\mathcal{K}}$-increasing and continuous,*
(2) *for all $i \leq \sigma$, $M_i \in \mathcal{K}^{am}_\mu$,*
(3) *for all $i < \sigma$, M_{i+1} is universal over M_i*
(4) $\mathrm{cf}(\sigma) = \sigma \leq \mu^+ \leq \lambda$ *and*
(5) $p \in$ ga-S(M_σ).

Then there exists $i < \sigma$ such that p does not μ-split over M_i.

We derive the following extension property for non-splitting types:

Theorem 7.3 (Extension of non-splitting types)**.** *Suppose that $M, \check{M} \in \mathcal{K}_\mu$ is universal over N and* ga-tp$(a/M, \check{M})$ *does not μ-split over N. Let $\check{M} \in \mathcal{K}_\mu$ be a (μ, μ^+)-limit containing $\bar{a} \bigcup M$.*

Let M' be an extension of M of cardinality μ. Then there exists a $\prec_{\mathcal{K}}$-mapping f such that $f : M' \to \check{M}$, $f \restriction M = id_M$ and ga-tp$(a/f(M'))$ *does not μ-split over N.*

Proof. Notice that $\check{M}$ is universal over M. So we may assume that $\check{M}$ contains M'. Since M is universal over N, there exists a $\prec_{\mathcal{K}}$ mapping $h' : M' \to M$ with $h' \restriction N = id_N$. Let $M'' \prec_{\mathcal{K}} \check{M}$ be an extension of M' such that h' can be extended to $h'' : M'' \cong M$. Further extend h'' to h so that $\mathrm{dom}(h) \prec_{\mathcal{K}} \check{M}$ and $a \in \mathrm{rge}(h)$. By invariance, ga-tp$(h^{-1}(a)/M'')$ does not μ-split over N.

Subclaim 7.4. ga-tp$(h^{-1}(a)/M) =$ ga-tp(a/M).

Proof. Let $N_1 := M''$ and $N_2 = M$. Let $p :=$ ga-tp$(h^{-1}(a)/M'')$. Consider the mapping $h : N_1 \cong N_2$. Since p does not μ-split over N, $h(p \restriction N_1) = p \restriction N_2$. Let us calculate this

$$h(p \restriction N_1) = \text{ga-tp}(h(h^{-1}(a))/g(M'') = \text{ga-tp}(a/M).$$

While,

$$p \restriction N_2 = \text{ga-tp}(h^{-1}(a)/M).$$

Thus ga-tp$(h^{-1}(a)/M) =$ ga-tp(a/M) as required. ⊣

From the subclaim, we can find a $\prec_{\mathcal{K}}$-mapping g and a model $M^* \prec_{\mathcal{K}} \check{M}$ such that $g : M'' \to M^*$, $g \restriction M = id_M$ and $g \circ h^{-1}(a) = a$. Notice that ga-tp$(a/g(M''), \check{M})$ does not μ-split over M. ⊣

Definition 7.5.

$$^{+}\mathcal{K}^*_{\mu,\alpha} := \left\{ (\bar{M}, \bar{a}, \bar{N}) \;\middle|\; \begin{array}{l} (\bar{M}, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}; \\ \bar{N} = \langle N_i \mid i+1 < \alpha \rangle; \\ \text{ for every } i+1 < \alpha, N_i \prec_{\mathcal{K}} M_i, ; \\ M_i \text{ is universal over } N_i \text{ and;} \\ \text{ga-tp}(a_i, M_i, M_{i+1}) \text{ does not } \mu\text{-split over } N_i. \end{array} \right\}$$

Similar to the case of $\mathcal{K}^*_{\mu,\alpha}$ we define an ordering,

Definition 7.6. For $(\bar{M}, \bar{a}, \bar{N})$ and $(\bar{M}', \bar{a}', \bar{N}') \in {}^{+}\mathcal{K}^*_{\mu,\alpha}$, we say $(\bar{M}, \bar{a}, \bar{N}) <^c_{\mu,\alpha} (\bar{M}', \bar{a}', \bar{N}')$ iff

(1) $(\bar{M}, \bar{a}) <^b_{\mu,\alpha} (\bar{M}', \bar{a}')$
(2) $\bar{N} = \bar{N}'$ and
(3) for every $i < \alpha$, ga-tp$(a_i/M'_{i+1}, M'_{i+1})$ does not μ-split over N_i.

Notation 7.7. We say that $(\bar{M}, \bar{a}, \bar{N})$ is *nice* iff when i is a limit ordinal $\bigcup_{j<i} M_j$ is an amalgamation base.

The following theorem is a partial solution to a problem from [ShVi]:

Theorem 7.8 (The $<^c_{\mu,\alpha}$-extension property for nice towers)**.** *If $(\bar{M}, \bar{a}, \bar{N}) \in {}^{+}\mathcal{K}^*_{\mu,\alpha}$ is nice, then there exists a nice $(\bar{M}', \bar{a}, \bar{N}') \in {}^{+}\mathcal{K}^*_{\mu,\alpha}$ such that $(\bar{M}, \bar{a}, \bar{N}) <^c_{\mu,\alpha} (\bar{M}', \bar{a}, \bar{N}')$.*

Proof. Let μ be a cardinal and α a limit ordinal such that $\alpha < \mu^+ \leq \lambda$. Let $(\bar{M}, \bar{a}, \bar{N}) \in {}^+\mathcal{K}^*_{\mu,\alpha}$ be given. Denote by M_α a model in $\mathcal{K}^{am}_\mu$ extending $\bigcup_{i<\alpha} M_i$. Fix $\check{M}$ to be a (μ, μ^+)-limit model over M_α.

We will define by induction on $i < \alpha$ a sequence of models $\langle M'_i \mid i < \alpha\rangle$ and a sequences of $\prec_{\mathcal{K}}$-mappings, $\langle f'_{j,i} \mid j < i < \alpha\rangle$ and $\langle \check{f}_{j,i} \mid j < i < \alpha\rangle$ such that for $i \leq \alpha$:

(1) $(\langle f'_{j,i}(M'_j) \mid j \leq i\rangle, \bar{a} \restriction i, \bar{N} \restriction i)$ is a $<^c_{\mu,i}$ of $(\bar{M}, \bar{a}, \bar{N}) \restriction i$,
(2) $(\langle M_j \mid j < i\rangle, \langle f'_{j,i} \mid j \leq i\rangle)$ forms a directed system,
(3) M'_i is universal over M_i,
(4) M'_{i+1} is universal over $f'_{i,i+1}(M'_i)$,
(5) $f'_{j,i} \restriction M_j = id_{M_j}$,
(6) $M'_i \prec_{\mathcal{K}} \check{M}$,
(7) $f_{j,i}$ can be extended to an autmorphism of $\check{M}$, $\check{f}_{j,i}$, for $j \leq i$ and
(8) $(\langle \check{M} \mid j < i\rangle, \langle \check{f}_{k,j} \mid k \leq j < i\rangle)$ forms a directed system.

The construction is enough: We can take M'_α and $\langle f'_{i,\alpha} \mid i < \alpha\rangle$ to be the direct limit of $(\langle M'_i \mid i < \alpha\rangle, \langle f'_{j,i} \mid j \leq i < \alpha\rangle)$. Since $f'_{j,i} \restriction M_j = id_{M_j}$, for every $j \leq i < \alpha$, we may assume that $f'_{i,\alpha} \restriction M_i = id_{M_i}$ for every $i < \alpha$. Notice that $(\langle f'_{i,\alpha}(M'_i) \mid i < \alpha\rangle, \bar{a})$ is a $<^c_{\mu,\alpha}$-extension of $(\bar{M}, \bar{a})$.

The construction is possible:

$i = 0$: Since M_0 is an amalgamation base, we can find $M''_0 \in \mathcal{K}^*_\mu$ (a first approximation of the desired M'_0) such that M''_0 is universal over M_0. By Theorem 7.3, we may assume that ga-tp(a_0/M''_0) does not μ-split over N_0 and $M''_0 \prec_{\mathcal{K}} \check{M}$. Since $a_0 \notin M_0$ and ga-tp(a_0/M_0) does not μ-split over N_0, we know that $a_0 \notin M''_0$. But, we might have that for some $l > 0$, $a_l \in M''_0$. We use weak disjoint amalgamation to avoid $\{a_l \mid 0 < l < \alpha\}$. By the Downward Löweneim-Skolem Axiom for AECs (Axiom 6) we can choose $M^2 \in \mathcal{K}_\mu$ such that $M''0$, $M_1 \prec_{\mathcal{K}} M^2 \prec_{\mathcal{K}} \check{M}$.

By Corollary 5.3 (applied to M_1, M_α, M^2 and $\langle a_l \mid 0 < l < \alpha\rangle$), we can find a $\prec_{\mathcal{K}}$-mapping f such that

· $f : M^2 \to \check{M}$
· $f \restriction M_1 = id_{M_1}$
· $f(M^2) \cap \{a_l \mid 0 < l < \alpha\} = \emptyset$

Define $M'_0 := f(M''_0)$. Notice that $a_0 \notin M'_0$ because $a_0 \notin M''_0$ and $f(a_0) = a_0$. Clearly $M'_0 \cap \{a_l \mid 0 \leq l < \alpha\} = \emptyset$. We need only verify that ga-tp(a_0/M'_0) does not μ-split over N_0. By invariaince, ga-tp(a_0/M''_0) does not μ-split over N_0 implies that ga-tp$(f(a_0)/f(M''_0))$ does not μ-split over N_0. But recall $f(a_0) = a_0$ and $f(M''_0) = M'_0$. Thus ga-tp(a_0/M'_0) does not μ-split over N_0.

Set $\check{f}_{0,0} := id_{\check{M}}$ and $f'_{0,0} := id_{M'_0}$.

$i = j + 1$: Suppose that we have completed the construction of all $k \leq j$. Since $M'_j, M_{j+1} \prec_{\mathcal{K}} \check{M}$, we can apply the Downward-Löwenheim Axiom for AECs to find M'''_{j+1} (a first approximation to M'_{j+1}) a model of cardinality μ extending both M'_j and M_{j+1}. WLOG we may assume that M'''_{j+1} is a limit model of cardinality μ and M'''_{j+1} is universal over M_{j+1} and M'_j. By Theorem 7.3, we can find a $\prec_{\mathcal{K}}$ mapping $f : M'''_{j+1} \to \check{M}$ such that $f \restriction M_{j+1} = id_{M_{j+1}}$ and ga-tp$(a_{j+1}/f(M'''_{j+1}))$ does not μ-split over N_{j+1}. Set $M''_{j+1} := f(M'''_{j+1})$.

Subclaim 7.9. $a_{j+1} \notin M''_{j+1}$

Proof. Suppose that $a_{j+1} \in M''_{j+1}$. Since M'_{j+1} is universal over N_{j+1}, there exists a $\prec_{\mathcal{K}}$-mapping, $g : M''_{j+1} \to M_{j+1}$ such that $g \restriction N_{j+1} = id_{N_{j+1}}$. Since ga-tp$(a_{j+1}/M''_{j+1})$ does not μ-split over N_{j+1}, we have that

$$\text{ga-tp}(a_{j+1}/g(M''_{j+1}) = \text{ga-tp}(g(a_{j+1})/g(M''_{j+1}).$$

Notice that because $g(a_{j+1}) \in g(M''_{j+1})$, we have that $a_{j+1} \in g(M''_{j+1})$. But $g(M''_{j+1}) \prec_{\mathcal{K}} M_{j+1}$. This contradicts the definition of towers: $a_{j+1} \notin M_{j+1}$. $\dashv$

M''_{j+1} may serve us well if it does not contain any a_l for $j+1 \leq l < \alpha$, but this is not guaranteed. So we need to make an adjustment. Let M^2 be a model of cardinality μ such that $M_{j+2}, M''_{j+1} \prec_{\mathcal{K}} M^2 \prec_{\mathcal{K}} \check{M}$. Notice that $\check{M}$ is universal over M_{j+2}. Thus we can apply Corollary 5.3 to M_{j+2}, M_α, M^2 and $\langle a_l \mid j+2 \leq l < \alpha\rangle$. This yields a $\prec_{\mathcal{K}}$-mapping h such that

- $h : M^2 \to \check{M}$
- $h \restriction M_{j+2} = id_{M_{j+2}}$ and
- $h(M^2) \cap \{a_l \mid j+2 \leq l < \alpha\} = \emptyset$.

Set $M'_{j+1} := h(M''_{j+1})$. Notice that by invariance, ga-tp(a_{j+1}/M''_{j+1}) does not μ-split over N_{j+1} implies that ga-tp$(h(a_{j+1})/h(M''_{j+1})$ does not μ-split over $h(N_{j+1})$. Recalling that $h \restriction M_{j+2} = id_{M_{j+2}}$ we have that ga-tp(a_{j+1}/M''_{j+1}) does not μ-split over N_{j+1}. We need to verify that $a_{j+1} \notin M'_{j+1}$. This holds because $a_{j+1} \notin M''_{j+1}$ and $h(a_{j+1}) = a_{j+1}$.

Set $f'_{j+1,j+1} = id_{M_{j+1,j+1}}$ and $\check{f}_{j+1,j+1} = id_{\check{M}}$ and $f'_{j,j+1} := h \circ f \restriction M'_j$. Since $\check{M}$ is a (μ,μ^+)-limit over both M'_j and $f'_{j,j+1}(M'_j)$, we can extend $f'_{j,j+1}$ to an automorphism of $\check{M}$, denoted by $\check{f}_{j,j+1}$.

To guarantee that we have a directed system, for $k < j$, define $f'_{k,j+1} := f'_{j,j+1} \circ f'_{k,j}$ and $\check{f}_{k,j+1} := \check{f}_{j,j+1} \circ \check{f}_{k,j}$.

i is a limit ordinal: Suppose that $(\langle M'_j \mid j < i\rangle, \langle f'_{k,j} \mid k \leq j < i\rangle)$ and $(\langle \check{M} \mid j < i\rangle, \langle \check{f}_{k,j} \mid k \leq j < i\rangle)$ have been defined. Since they are both directed systems, we can take direct limits. By Claim 6.11, we may assume that $(M^*_i, \langle f^*_{j,i} \mid j < i\rangle)$ and $(\check{M}, \langle \check{f}^*_{j,i} \mid j < i\rangle)$ are the respective direct limits such that $M^*_i \prec_{\mathcal{K}} \check{M}$ and $\bigcup_{j<i} M_j \prec_{\mathcal{K}} M^*_i$. By Condition (4) of the construction, notice that M^*_i is a (μ, i)-limit model witnessed by $\langle f^*_{j,i}(M'_j) \mid j < i\rangle$. Hence M^*_i is an amalgamation base. Since M^*_i and M_i both live inside of $\check{M}$, we can find $M'''_i \in \mathcal{K}^*_\mu$ which is universal over M_i and universal over M^*_i.

By Theorem 7.3 we can find a $\prec_{\mathcal{K}}$-mapping $f : M'''_i \to \check{M}$ such that $f \restriction M_i = id_{M_i}$ and ga-tp$(a_i/f(M'''_i))$ does not μ-split over N_i. Set $M''_i := f(M'''_i)$. By a similar argument to Subclaim 7, we can see that $a_i \notin M''_i$.

M''_i may contain some a_l when $i \leq l < \alpha$. We need to make an adjustment using weak disjoint amalgamtion. Let M^2 be a model of cardinality μ such that $M''_i, M_{i+1} \prec_{\mathcal{K}} M^2 \prec_{\mathcal{K}} \check{M}$. By Corollary 5.3 applied to M_i, M_α, M^2 and $\langle a_l \mid i < l < \alpha\rangle$ we can find $h : M''_i \to \check{M}$ such that $h \restriction M_{i+1} = id_{M_{i+1}}$ and $h(M^2) \cap \{a_l \mid i < l < \alpha\} = \emptyset$.

Set $M'_i := h(M''_i)$. We need to verify that $a_i \notin M'_i$ and ga-tp(a_i/M'_i) does not μ-split over N_i. Since $a_i \notin M''_i$ and $h(a_i) = a_i$, we have that $a_i \notin h(M''_i) = M'_i$. By invariance of non-splitting, ga-tp(a_i/M''_i) not μ-splitting over N_i implies that ga-tp$(h(a_i)/h(M''_i))$ does not μ-split over $h(N_i)$. Recalling our definition of h and M'_i. This yields ga-tp(a_i/M'_i) does not μ-split over N_i.

Set $f'_{i,i} := id_{M_i,i}$, $\check{f}_{i,i} := id_{\check{M}}$ and for $j < i$, $f'_{j,i} := h \circ f \circ f^*_{j,i}$.

Notice that for every $j < i$, $\check{M}$ is a (μ, μ^+)-limit over both M'_j and $f'_{j,i}(M'_j)$. Thus by the uniqueness of (μ, μ^+)-limit models, we can extend $f'_{j,i}$ to an automorphism of $\check{M}$, denoted by $\check{f}_{j,i}$.

$\dashv$

8. Uniqueness of Limit Models

Recall the running assumptions:

(1) $\mathcal{K}$ is an abstract elementary class,
(2) $\mathcal{K}$ has no maximal models,
(3) $\mathcal{K}$ is categorical in some $\lambda > LS(\mathcal{K})$,
(4) GCH and $\Phi_{\mu^+}(S^{\mu^+}_\theta)$ holds for every cardinal $\mu < \lambda$ and every regular θ with $\theta < \mu^+$.

Using Theorem 7.8, arguments as in [ShVi] and some new results we can conclude the uniqueness of limit models. This is a partial solution to a conjecture from [ShVi].

Theorem 8.1 (Uniqueness of Limit Models)**.** *Suppose $\mathcal{K}^{am}_\mu$ is closed under unions of length $< \mu^+$. Let μ be a cardinal θ_1, θ_2 limit ordinals such that $\theta_1, \theta_2 < \mu^+ \leq \lambda$. If M_1 and M_2 are (μ, θ_1) and (μ, θ_2) limit models over M, respectively, then there exists an isomorphism $f : M_1 \cong M_2$ such that $f \restriction M = id_M$.*

We begin working towards the theorem by introducing a more general notion of towers:

Definition 8.2 (Definition 3.3.1 of [ShVi])**.** For $\mathfrak{U}$ a set of intervals of ordinals $< \mu^+$, let

$$^+\mathcal{K}^*_{\mu,\mathfrak{U}} := \left\{ (\bar{M}, \bar{a}, \bar{N}) \;\middle|\; \begin{array}{l} \bar{M} = \langle M_i \mid i \in u \text{ for some interval } u \in \mathfrak{U}\rangle; \\ \bar{M} \text{ is } \prec_{\mathcal{K}} \text{ increasing, but not} \\ \text{necessarily continuous;} \\ a_i \in M_{i+1} \backslash M_i \text{ when } i, i+1 \in \bigcup \mathfrak{U}; \\ \bar{N} = \langle N_i \mid i \in \bigcup \mathfrak{U}\rangle; \\ M_i \text{ is universal over } N_i \text{ when } i \in \bigcup \mathfrak{U} \text{ and} \\ \text{ga-tp}(a_i, M_i, M_{i+1}) \text{ does not } \mu-\text{split over } N_i \end{array} \right\}$$

Notice that these *scattered towers* are in some sense subtowers of the towers $^+\mathcal{K}^*_{\mu,\alpha}$. Hence we can consider the restriction of $<^c_{\mu,\alpha}$ to this class:

Definition 8.3 (Definition 3.3.2 of [ShVi])**.** Let $\mathfrak{U}_1, \mathfrak{U}_2$ be such that $\bigcup \mathfrak{U}_1 \subseteq \bigcup \mathfrak{U}_2$. Let $(\bar{M}^l, \bar{a}^l, \bar{N}^l) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}_l}$ for $l = 1, 2$. $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \leq^c (\bar{M}^2, \bar{a}^2, \bar{N}^2)$ iff for $i \in \bigcup \mathfrak{U}_1$,

(1) $M^1_i \preceq_{\mathcal{K}} M^2_i$, $a^1_i = a^2_i$ and $N^1_i = N^2_i$ and
(2) if $M^1_i \neq M^2_i$, then M^2_i is universal over M^1_i.

We say that $(\bar{M}^1, \bar{a}^1, \bar{N}^1) <^c (\bar{M}^2, \bar{a}^2, \bar{N}^2)$ provided that for every $i \in \bigcup \mathfrak{U}_1$, $M^1_i \neq M^2_i$.

We first need a generalization of the $<^c_{\mu,\alpha}$-extension property:

Definition 8.4. A scattered tower $(\bar{M}, \bar{a}, \bar{N}) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}}$ is said to be *nice* provided that whenever $i \in \bigcup \mathfrak{U}$ is a limit of some sequence from $\mathfrak{U}$, then $\bigcup_{j \in \mathfrak{U},\, j<i} M_j$ is an amalgamation base.

Theorem 8.5 ($<^c$-Extension Property for Nice Scattered Towers)**.** *Let* $\mathfrak{U}^1$, $\mathfrak{U}^2$ *be such that* $\bigcup \mathfrak{U}^1 \subseteq \bigcup \mathfrak{U}^2$. *Let* $(\bar{M}^1, \bar{a}^1, \bar{N}^1) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}^1}$ *be a nice scattered tower. There exists a nice scattered tower* $(\bar{M}^2, \bar{a}^2, \bar{N}^2) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}^2}$ *such that* $(\bar{M}^1, \bar{a}^1, \bar{N}^1) <^c (\bar{M}^2, \bar{a}^2, \bar{N}^2)$.

Proof. WLOG we can rewrite $\mathfrak{U}^1$ and $\mathfrak{U}^2$ as collections of disjoint intervals. Moreover, we can write $\mathfrak{U}^1 := \{u^1_i \mid i < \alpha\}$ where $\sup\{u^1_i\} < \min\{u^1_{i+1}\}$. We then can enumerate $\mathfrak{U}^2$ conveniently as $\{u^2_{i,j} \mid i < \alpha,\ j < |\mathfrak{U}^2|\}$ such that $u^2_{i,0}$ is an end-extension of u^1_i and for $j > 0$,

$$u^2_{i,j} := \begin{cases} u & ;\text{ when the number of intervals of } \mathfrak{U}^2 \text{ between} \\ & \ \ u^1_i \text{ and } u^1_{i+1} \text{ is } \geq j \text{ and } u \text{ is the } j^{th} \text{ interval of} \\ & \ \ \mathfrak{U}^2 \text{ between } u^1_i \text{ and } u^1_{i+1} \\ \emptyset & ;\text{otherwise.} \end{cases}$$

Given $(\bar{M}^1, \bar{a}, \bar{N}) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}^1}$ a nice tower, we will find a $<^c$-extension in $^+\mathcal{K}^*_{\mu,\mathfrak{U}^2}$ by using direct limits inside a (μ, μ^+)-limit model as we have done in the proofs of Theorem 6.7 and Theorem 7.8. We will leave out some details so as not to be bogged down with notation. The details we hide will be no different than those already described in the proofs of Theorem 6.7 and

Theorem 7.8. As before, fix $\check{M}$ a (μ,μ^+)-limit model containing $\bigcup_{t\in\bigcup\mathfrak{U}^1} M^1_t$. We will demonstrate how to find an extension of $(\bar M^1,\bar a,\bar N)\restriction u^1_0$ to $\{u^2_{0,j}\mid j<|\mathfrak{U}^2|\|$ such that the extension lies inside of $\check{M}$ and contains no illegal a_i.

By Theorem 7.8, we can find a $<^c$-extension of $(\bar M^1,\bar a,\bar N)\restriction u^1_0$, say $(\bar M',\bar a,\bar N)\in\ ^+\mathcal{K}^*_{\mu,\{u^1_0\}}$, such that $\bar M'$ lies inside of $\check{M}$ and avoids the bad a_i's. By Weak Disjoint Amalgamation (Corollary 5.3) and niceness, we can find an extension of $M^1_{\min\{u^1_1\}}$, denoted by $M^2_{u_{1,0}}$ such that

(1) $M^2_{\min\{u_{1,0}\}}$ avoids the bad a_i,
(2) $M^2_{\min\{u_{1,0}\}}$ is universal over $M^1_{\min\{u_{1,0}\}}$
(3) $M^2_{\min\{u_{1,0}\}}$ is a $(\mu,|\mathfrak{U}^2|)$-limit over $\bigcup_{t\in u^1_0} M^1_t$ witnessed by $\langle M^*_j\mid j<|\mathfrak{U}^2|\rangle$ where each M^*_j is a limit model over $\bigcup_{t\in u^1_0} M^1_t$ as well,
(4) $M^2_{\min\{u_{1,0}\}}\prec_{\mathcal{K}}\check{M}$ and
(5) $M^2_{\min\{u_{1,0}\}}$ contains an image of $\bar M'$ fixing $\bar M^1\restriction u^1_0$.

Set M^2_t to be the image of M'_t for $t\in u^1_0$. For $t\in u^2_{0,j}$, if t is the j^{th}-element of $\mathfrak{U}^2$ which lies between $\sup\{u^1_0\}$ and $\min\{u^1_1\}$, then set $M^2_t:=M^*_{j+1}$. For each such $t\in u^2_{0,j}$, we can fix $a_t\in M^2_{t+1}\backslash M^2_t$. Since M^2_t is a limit model over $\bigcup_{s\in u^1_0} M^1_s$, there exists $s\in u^1_0$ such that ga-tp(a_t/M^2_t) does not μ-split over M^1_s and M^2_t is universal over M^1_s. Thus we can set $N^2_t:=M^1_s$. This completes the definition of a $<^c$-extension of $(\bar M^1,\bar a,\bar N)\restriction u^1_0$ to $\{u^2_{0,j}\mid j<|\mathfrak{U}^2|\}$ inside $\check{M}$.

We can now apply an argument as in the successor case of Theorem 7.8 to first find an extension of $(\bar M^1,\bar a,\bar N)\restriction\{u^1_0,u^1_1\}$ to $\{u^2_{0,j}\mid j<|\mathfrak{U}^2|\}\bigcup u^2_{1,0}$ inside of $\check{M}$ avoiding the bad a'_is and containing an image of our previous work. We then can proceed as in the base case above to further extend $(\bar M^1,\bar a,\bar N)\restriction\{u^1_0,u^1_1\}$ to a tower with index set $\{u^1_{i,j}\mid i\in\{0,1\},j<|\mathfrak{U}^2|\}$ inside $\check{M}$.

This construction is done in a coherent manner so that we can take direct limits in limit stages as we did in the proof of Theorem 7.8.

⊣

The following concept was introduced in [ShVi] for $^+\mathcal{K}^*_{\mu,\alpha}$ towers.

Definition 8.6. A tower $(\bar M,\bar a,\bar N)\in^+\mathcal{K}^*_{\mu,\mathfrak{U}}$ is said to be *reduced* provided that for every $(\bar M',\bar a,\bar N)\in\ ^+\mathcal{K}^*_{\mu,\mathfrak{U}}$ with $(\bar M,\bar a,\bar N)\leq^c(\bar M',\bar a,\bar N)$ we have that for every $i\in\bigcup\mathfrak{U}$,

$$M'_i\cap\bigcup_{j\in\bigcup\mathfrak{U}} M_j=M_i.$$

Notice that for a tower in $^+\mathcal{K}^*_{\mu,\mathfrak{U}}$ to be reduced, it only depends on other towers with the *same* index set. However, if we take a $<^c$-increasing and continuous chain of reduced towers with increasing index sets, the union will be reduced:

Proposition 8.7. *Let* $\langle \mathfrak{U}_\gamma \mid \gamma < \beta\rangle$ *be an increasing and continuous sequence of sets of intervals. If* $\langle (\bar{M}, \bar{a}, \bar{N})^\gamma \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}_\gamma} \mid \gamma < \beta\rangle$ *is* $<^c$*-increasing and continuous sequence of reduced towers, then the union of these towers is reduced.*

Proof. Denote by $(\bar{M}, \bar{a}, \bar{N})^\beta$ the limit of the sequence of towers and $\mathfrak{U}_\beta$ the limit of the intervals. Suppose that it is not reduced. Let $(\bar{M}', \bar{a}, \bar{N}) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}_\beta}$ witness this. Then there exists an $i \in \bigcup \mathfrak{U}_\beta$ and an element a such that $a \in (M'_i \cap \bigcup_{j\in\mathfrak{U}_\beta} M^\beta_j)\backslash M^\beta_i$. There exists $\gamma < \beta$ such that $i \in \mathfrak{U}_\gamma$ and there exists $j \in \mathfrak{U}_\gamma$ such that $a \in M^\gamma_j$. Now consider the tower in ${}^+\mathcal{K}^*_{\mu,\mathfrak{U}_\gamma}$, $(\bar{M}', \bar{a}, \bar{N}) \restriction \mathfrak{U}_\gamma$. Notice that $(\bar{M}', \bar{a}, \bar{N}) \restriction \mathfrak{U}_\gamma$ witnesses that $(\bar{M}, \bar{a}, \bar{N})^\gamma$ is not reduced. ⊣

This is where we need to introduce the additional assumption. It will allow us to take $<^c$-extensions of all scattered towers (not just the nice ones). For the remainder of the paper we assume that

Assumption 8.8. $\mathcal{K}^{am}_\mu$ is closed under unions of length μ^+.

The proofs of the following two results on reduced towers (Proposition 8.9 and Theorem 8.10) rely on the $<^c$-extension property. The proofs were outlined in [ShVi] for the particular case of $\mathfrak{U} = \{[0, \alpha)\}$ when α is a limit ordinal $< \mu^+$ (see Fact 3.1.13 and Theorem 3.1.15 of their work). However, Shelah and Villaveces were unable to prove the $<^c$-extension property. The proofs of Proposition 8.9 and Theorem 8.10 for scattered towers are the same as for ${}^+\mathcal{K}^*_{\mu,\alpha}$-towers.

Proposition 8.9 (Density of reduced towers)**.** *Let* $(\bar{M}, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ *be given. Then there exists* $(\bar{M}', \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ *such that*

- $(\bar{M}, \bar{a}, \bar{N}) <^c (\bar{M}', \bar{a}, \bar{N})$ *and*
- $(\bar{M}', \bar{a}, \bar{N})$ *is reduced.*

Theorem 8.10 (Reduced towers are continuous)**.** *If* $(\bar{M}, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ *is reduced, then* $\bar{M}$ *is continuous.*

Definition 8.11 (Definition 3.2.1 of [ShVi])**.** For M a (μ, θ)-limit model,

(1) Let

$$\mathfrak{S}\mathrm{t}(M) := \left\{ (p, N) \;\middle|\; \begin{array}{l} N \prec_{\mathcal{K}} M; \\ N \text{ is a } (\mu, \theta)-\text{limit model}; \\ M \text{ is universal over } N \text{ and} \\ p \in \text{ga-S}(M) \text{ does not } \mu-\text{split over } N. \end{array} \right\}$$

and

(2) For types $(p_l, N_l) \in \mathfrak{St}(M)$ $(l = 1, 2)$, we say $(p_1, N_1) \sim (p_2, N_2)$ iff for every $M' \in \mathcal{K}^{am}_\mu$ extending M there is a $q \in S(M')$ extending both p_1 and p_2 such that q does not μ-split over N_1 and q does not μ-split over N_2.

Notice that $\sim$ is an equivalence relation on $\mathfrak{St}(M)$. This gives us another notion of types.

By Fact 2.19, we have

Fact 8.12. *For $M \in \mathcal{K}^{am}_\mu$, $|\,\mathfrak{St}(M)/\sim\,| \leq \mu$.*

We can then consider towers which are saturated with respect to $\mathfrak{St}(M)$:

Definition 8.13. A tower $(\bar{M}, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ is said to be *full* iff

(1) μ divides $\mathrm{cf}(\sup\{\bigcup \mathfrak{U}\})$ and
(2) if $\beta \in \bigcup \mathfrak{U}$ and $(p, N^*) \in \mathfrak{St}(M_\beta)$, then for some $i < \mu$ with $\beta + i \in \bigcup \mathfrak{U}$, we have that $(\text{ga-tp}(a_{\beta+1}, M_{\beta+i}, M_{\beta+i+1}), N_{\beta+i}) \sim (p, N^*)$.

Remark 8.14. Definition 8.13 appears in [ShVi] for the special case when $\mathfrak{U} = \{[0, \alpha)\}$ for α a limit ordinal $< \mu^+$ (see Definition 3.2.3 of their paper).

The following theorem is proved in [ShVi] under the particular instance of $\mathfrak{U} = \{[0, \alpha)\}$ for α a limit ordinal $< \mu^+$ (Theorem 3.2.4 of their work). We require the more general result, but the proof is similar to Shelah and Villaveces' argument.

Theorem 8.15. *If $(\bar{M}, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}}$ is full and $\bar{M}$ is continuous, then $\bigcup_{i \in \bigcup \mathfrak{U}} M_i$ is a $(\mu, \mathrm{cf}(\sup\{\bigcup \mathfrak{U}\}))$-limit model over M_0.*

In addition to these generalized results we need the following new theorem which is an analog to the statement that the union of $\kappa(T)$-many saturated models is saturated in first order stable theories. We are not implying that fullness is equivalent to saturation, but that the spirit of the results is similar. The following theorem was not stated in [ShVi] and is new:

Theorem 8.16 (Union of Full Towers is Full)**.** *Let α be a limit ordinal $<$ μ^+ and let $\mathfrak{U}$ be set of intervals such that $|\,\mathfrak{U}\,| < \mu^+$. If $\langle(\bar{M}^\beta, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}} \mid \beta < \alpha\rangle$ is a $<^c$-increasing and continuous chain of full towers for $\alpha < \mu^+$, then the union is a full tower.*

Proof. Let $\langle(\bar{M}^\beta, \bar{a}, \bar{N}) \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}} \mid \beta < \alpha\rangle$ be a $<^c$-increasing and continuous chain of towers. We need to verify that for $i \in \mathfrak{U}$ and $(p, N) \in \mathfrak{St}(\bigcup_{\beta<\alpha} M^\beta_i)$, that there exists $j < \mu$ such that $i + j \in \mathfrak{U}$ and $(p, N) \sim (\text{ga-tp}(a_{i+j}, \bigcup_{\beta<\alpha} M^\beta_{i+j}), N_{i+j})$.

By the definition of $<^c$, we have that $\bigcup_{\beta<\alpha} M^\beta_i$ is a (μ, α)-limit witnessed by $\langle M^\beta_i \mid \beta < \alpha\rangle$. By Theorem 7.2, there exists $\beta < \alpha$ such that p does not μ-split over M^β_i. Thus $(p \restriction M^{\beta+1}_i, M^\beta_i) \in \mathfrak{St}(M^{\beta+1}_i)$. By the assumption of fullness of the $\beta + 1^{st}$ tower, there exists a $j < \mu$ such that

$$(p \restriction M^{\beta+1}_i, M^\beta_i) \sim (\text{ga-tp}(a_{i+j}/M^{\beta+1}_{i+j}), N_{i+j}).$$

Recalling the definition of $\sim$, we know that there exists $q \in$ ga-S$(\bigcup_{\gamma<\alpha} M^{\gamma}_{i+j})$ such that

- $p \upharpoonright M^{\beta}_i \subseteq q$
- ga-tp$(a_{i+j}/M^{\beta+1}_{i+j}) \subseteq q$
- q does not μ-split over M^{β}_i and
- q does not μ-split over N_{i+j}.

Notice that it suffices to show

Subclaim 8.17. $(p, N) \sim ($ga-tp$(a_{i+j}/\bigcup_{\gamma<\alpha} M^{\gamma}_{i+j}), N_{i+j})$.

Proof of Subclaim 8.17. By definition of $\sim$, we have that

$$(p \upharpoonright M^{\beta+1}_i, M^{\beta}_i) \sim (p, N).$$

Recalling that ga-tp$(a_{i+j}/\bigcup_{\gamma<\alpha} M^{\gamma}_{i+j})$ does not μ-split over N_{i+j}, we see that

$$(\text{ga-tp}(a_{i+j}/M^{\beta+1}_{i+j}), N_{i+j}) \sim (\text{ga-tp}(a_{i+j}/\bigcup_{\gamma<\alpha} M^{\gamma}_{i+j}), N_{i+j}).$$

Since $\sim$ is transitive, we have that $(p, N) \sim ($ga-tp$(a_{i+j}/\bigcup_{\gamma<\alpha} M^{\gamma}_{i+j}), N_{i+j})$. $\dashv$

$\dashv$

Now we are ready to prove Theorem 8.1.

Proof of Theorem 8.1. WLOG θ_1 and θ_2 are regular. By Proposition 2.27 it suffices to construct a model which is simultaneously a (μ, θ_1)-limit model and a (μ, θ_2)-limit model. The idea is to build a (scattered) array of models such that at some point in the array, we will find a model which is a (μ, θ_1)-limit model witnessed by its height in the array and is a (μ, θ_2)-limit model witnessed by its horizontal position in the array. We will define a chain of scattered towers of length $\mu^+ \times \theta_1$ while increasing the index set of the towers as we proceed.

We will consider the index set $\mathfrak{U}^{(\alpha,\zeta)}$ at stage $(\alpha, \zeta) \in \mu^+ \times \theta_1$ where

$$\mathfrak{U}^{(\alpha,\zeta)} := \bigcup\{[\beta\mu, \beta\mu\theta_1 + \mu\zeta) \mid \beta < \alpha\}.$$

Define by induction on $(\alpha, \zeta) \in \mu^+ \times \theta_1$ the $<^c$-increasing and continuous sequence of scattered towers, $\langle (\bar{M}, \bar{a}, \bar{N})^{(\alpha,\zeta)} \in^+ \mathcal{K}^*_{\mu,\mathfrak{U}^{(\alpha,\zeta)}} \mid (\alpha, \zeta) \in \mu^+ \times \theta_1\rangle$, such that

(1) $(\bar{M}, \bar{a}, \bar{N})^{(\alpha,\zeta)}$ is reduced,
(2) $(\bar{M}, \bar{a}, \bar{N})^{(\alpha+1,0)} := \bigcup_{\zeta<\theta_1}(\bar{M}, \bar{a}, \bar{N})^{(\alpha,\zeta)}$ and
(3) in successor stages in new intervals of length μ put in representatives of all $\mathfrak{St}$-types from the previous stages.

This construction is possible by Theorem 8.5, Theorem 8.9, Theorem 8.10 and Fact 8.12.

Consider the mapping $f : \mu^+ \to \mu^+$ defined by

$$f(\alpha) := \min \left\{ \gamma \;\middle|\; \begin{array}{l} \text{for every } \beta < \alpha,\ i \in \bigcup \mathfrak{U}_\beta \text{ and} \\ \text{for every } (p, N) \in \mathfrak{St}(M_i^\beta) \text{ there} \\ \text{exists } \beta' < \gamma \text{ and } j < \mu \text{ such that} \\ (\text{ga-tp}(a_{i+j}/M_{i+j}^{\beta'}), N_{i+j}) \sim (p, N) \end{array} \right\}$$

By condition (3) of the construction, f can be defined. Then there exists a club C such that

$$\delta \in C \Rightarrow f \restriction \delta : \delta \to \delta.$$

Notice that by the definition of f, this implies

$$\delta \in C \Rightarrow (\bar{M}, \bar{a}, \bar{N})^{(\delta,0)} \text{ is full.}$$

Pick $\alpha \in C \cap S^{\mu^+}_{\theta_2}$.

Subclaim 8.18. *We can find $\langle \alpha_\zeta \mid \zeta < \theta_1 \rangle$ increasing and continuous such that $(\bar{M}, \bar{a}, \bar{N})^{(\alpha_\zeta,0)} \restriction \mathfrak{U}^{(\alpha,0)}$ is full.*

Proof of Subclaim. By Theorem 8.16. $\dashv$

Take such a sequence. We will see that

$$M^* := \bigcup_{\zeta<\theta_1} \bigcup_{i \in \bigcup \mathfrak{U}^{(\alpha,0)}} M_i^{(\alpha_\zeta,0)}$$

is a (μ, θ_1)-limit witnessed by $\langle \bigcup_{i\in\bigcup \mathfrak{U}^{(\alpha,0)}} M_i^{(\alpha_\zeta,0)} \mid \zeta < \theta_1 \rangle$. Notice that by Theorem 8.16 M^* is full. Furthermore, we see that M^* is continuous since it is reduced. Now we can apply Theorem 8.15 to conclude that M^* is a $(\mu, \mathrm{cf}(\sup\{\bigcup \mathfrak{U}^{(\alpha,0)}\}))$-limit model. But by our choice of α, we have that $\mathrm{cf}(\sup\{\bigcup \mathfrak{U}^{(\alpha,0)}\}) = \theta_2$. Thus M^* is also a (μ, θ_2)-limit model.

$\dashv$

The above proof implicitly shows the existence of full towers:

Corollary 8.19. *There exists an interval $\mathfrak{U}$ and a tower $(\bar{M}, \bar{a}, \bar{N}) \in {}^+\mathcal{K}^*_{\mu,\mathfrak{U}}$ such that $(\bar{M}, \bar{a}, \bar{N})$ is full.*

9. Future Work

(1) Removing the extra assumption from Theorem 8.1 by proving either:

Conjecture 9.1 (Density of nice towers)**.** *For every $(\bar{M}, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$, there exists $(\bar{M}', \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ such that $(\bar{M}, \bar{a}) \leq^b_{\mu,\alpha} (\bar{M}', \bar{a})$ and $(\bar{M}', \bar{a})$ is nice.*

or

Conjecture 9.2. *If $(\bar{M}, \bar{a}) \in \mathcal{K}^*_{\mu,\alpha}$ is reduced, then $\bar{M}$ is continuous.*

(2) Deriving the amalgamation property and/or Conjecture 1.1 from Theorem 8.1.
(3) Analyzing the magnitude of extra assumption in Theorem 8.1.

Conjecture 9.3. *Extra assumption implies that all amalgamation bases are (μ, κ) amalgamation bases for $\mu \leq \kappa < \lambda$.*

Conjecture 9.4. *The categoricity model is weakly model homogeneous.*

Definition 9.5. N is *weakly model homoegeneous* iff for every $M \prec_{\mathcal{K}} N$ with $M \in \mathcal{K}^{am}_{<\|N\|}$ we have that N is universal over M.

Corollary 9.6 (to Conjecture 9.4)**.** *The categoricity model is Galois-saturated.*

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