Toward a Categoricity Theorem for Abstract Elementary Classes

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NON-ELEMENTARY CLASSES

There are many natural mathematical classes that cannot be captured by £rst order logic:

1. Archimedian ordered £elds

2. Locally £nite groups

3. Well-ordered sets

4. Noetherian Rings, etc.

EXTENSIONS OF FIRST-ORDER LOGIC

1. $L_{\kappa,\omega}$ when $\kappa > \omega$

2. $L(Q)$ when Q is interpretted as the Keisler-quanti£er (there exist at least $\aleph_1$ many)

3. $L_{\omega_1,\omega}(Q)$

De£nition. Let L_1 be an expansion of the language L. Suppose T_1 is a £rst order theory in L_1 and Γ is a collection of £rst order types without parameters in the language of L_1. We denote by

$$PC(T_1, \Gamma, L) :=$$

$\{(M \restriction L) \ : \ M \models T_1$ and M omits every type from $\Gamma\}$.

If $\mathcal{K}$ is a class of models such that $\mathcal{K} = PC(T_1, \Gamma, L(\mathcal{K}))$ for some T_1, Γ and L_1 with $\mu = |T_1|+|\Gamma|+|L_1|+\aleph_0$, then we call $\mathcal{K}$ a PC_μ-*class*.

ABSTRACT ELEMENTARY CLASSES

De£nition. Let $\mathcal{K}$ be a class of structures all in the same similarity type $L(\mathcal{K})$, and let $\prec_{\mathcal{K}}$ be a partial order on $\mathcal{K}$. The ordered pair $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an *abstract elementary class* iff

A0 (Closure under isomorphism)

(a) For every $M \in \mathcal{K}$ and every $L(\mathcal{K})$-structure N if $M \cong N$ then $N \in \mathcal{K}$.

(b) Let $N_1, N_2 \in \mathcal{K}$ and $M_1, M_2 \in \mathcal{K}$ such that there exist $f_l : N_l \cong M_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_{\mathcal{K}} N_2$ implies that $M_1 \prec_{\mathcal{K}} M_2$.

A1 For all $M, N \in \mathcal{K}$ if $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.

A2 Let M, N, M^* be $L(\mathcal{K})$-structures. If $M \subseteq N$, $M \prec_{\mathcal{K}} M^*$ and $N \prec_{\mathcal{K}} M^*$ then $M \prec_{\mathcal{K}} N$.

A3 (Löwenheim-Skolem) There exists a cardinal $LS(\mathcal{K}) \geq +\aleph_0 + |L(\mathcal{K})|$ such that for every $M \in \mathcal{K}$ and for every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $N \prec_{\mathcal{K}} M$, $|N| \supseteq A$ and $\|N\| \leq |A| + LS(\mathcal{K})$.

A4 (Tarski-Vaught Chain)

(a) For every regular cardinal μ and every $N \in \mathcal{K}$ if $\{M_i \prec_{\mathcal{K}} N \ : \ i < \mu\} \subseteq \mathcal{K}$ is increasing (i.e. $i < j \Longrightarrow M_i \prec_{\mathcal{K}} M_j$) then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $\bigcup_{i<\mu} M_i \prec_{\mathcal{K}} N$.

(b) For every regular μ, if $\{M_i \ : \ i < \mu\} \subseteq \mathcal{K}$ is increasing then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i<\mu} M_i$.

Shelah's Presentation Theorem. If $\mathcal{K}$ is an AEC, then $\mathcal{K}$ is a PC_μ-class for some $\mu \leq 2^{L(\mathcal{K})}$.

Notation.

1. For $M, N \in \mathcal{K}$ a monomorphism $f : M \to N$ is called a $\prec_{\mathcal{K}}$-*embedding* iff $f[M] \prec_{\mathcal{K}} N$. Denote this by $f : M \hookrightarrow_{\mathcal{K}} N$.

2. $\mathcal{K}_\lambda := \{M \in \mathcal{K} : \|M\| = \lambda\}$.

3. $\mathcal{K}_{<\lambda} := \{M \in \mathcal{K} : \|M\| < \lambda\}$.

In pure model theory, through solutions to §ỏs Conjecture, central concepts for the classi£cation of £rst order theories were identi£ed (ie. prime models, minimal types, super-stability, etc.).

§ỏs conjecture.(1956) Let T be a £rst-order theory. If there exists $\lambda > |T| + \aleph_0$ such that $I(\lambda, T) = 1$ then $I(\mu, T) = 1$ holds for every $\mu > |T| + \aleph_0$.

Our goal in studying AECs is to develop a classi£cation theory for non-elementary classes. A test question for the proposed stability theory was identi£ed by Shelah in the 70s for $L_{\omega_1,\omega}$. Later this conjecture was generalized to a categoricity transfer theorem for AECs.

Shelah's conjecture.(about 1977) Let $\mathcal{K}$ be an AEC. If $\mathcal{K}$ is categorical in some $\lambda \geq Hanf(\mathcal{K})$, then $\mathcal{K}$ is categorical in every $\mu \geq Hanf(\mathcal{K})$.

Status - very open. Partial results:

1. Keisler (1971), for $L_{\omega_1,\omega}$ under the additional assumption of existence sequentially homogeneous model.

2. Shelah (1984) [87a] and [87b] for excellent classes.

3. Lessmann (1998) for countable £nite diagrams.

4. Makkai and Shelah [Sh285] for $L_{\kappa,\omega}$-theories under the additional assumption that κ is a strongly compact cardinal, AND $\lambda = \chi^+$.

5. Kolman-Shelah [Sh362] and Shelah [Sh 472] partial going down results for $L_{\kappa,\omega}$-theories where κ is a measurable cardinal and $\lambda = \chi^+$.

6. Shelah [Sh394] partial going down result for AECs with the Amalgamation Property under the assumption that $\lambda = \chi^+$.

CONTEXT OF SHELAH AND VILLAVECES' WORK

1. GCH holds,
2. $\mathcal{K}$ is an abstract elementary class,
3. $\mathcal{K}$ has no maximal models and
4. $\mathcal{K}$ is categorical in some λ with $\lambda \geq LS(\mathcal{K})$.

The notion of Galois types

De£nition.

$$\mathcal{K}^3_\mu := \{(M,N,a) \mid a \in N;\ M \prec_{\mathcal{K}} N;\ M,N \in \mathcal{K}_\mu\}.$$

De£nition. Let $M \in \mathcal{K}_\mu$, We say that (M,N_1,a_1) and $(M,N_2,a_2) \in \mathcal{K}^3_\mu$, are $\sim$-*related*, written

$$(M,N_1,a_1) \sim (M,N_2,a_2),$$

if there exists $N \in \mathcal{K}_\lambda$ and $\prec_{\mathcal{K}}$-embeddings

$$h_1 : N_1 \to N \quad \text{and} \quad h_2 : N_2 \to N,$$

such that

$$h_1(a_1) = h_2(a_2)$$

and the following diagram commutes:

$$\begin{tikzcd} N_1 \arrow[r, "h_1"] & N \\ M \arrow[u, "id"] \arrow[r, "id"'] & N_2 \arrow[u, "h_2"'] \end{tikzcd}$$

AMALGAMATION PROPERTY

De£nition. Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an abstract elementary class, and let $\mu \geq \text{LS}(\mathcal{K})$ be a cardinal.

1. We say that $M_0 \in \mathcal{K}_\mu$ is an *amalgamation base* iff for every $M_1, M_2 \in \mathcal{K}_\mu$ and every $f_\ell : M_0 \hookrightarrow_{\mathcal{K}} M_\ell$ for $\ell = 1, 2$, there exists $N \in \mathcal{K}_\mu$ so that there are $g_1 : M_1 \hookrightarrow_{\mathcal{K}} N$ and $g_2 : M_2 \hookrightarrow_{\mathcal{K}} N$ so that $g_1 \circ f_1 \restriction |M_0| = g_2 \circ f_2 \restriction |M_0|$. Namely, the following diagram is commutative:

$$\begin{tikzcd} M_1 \arrow[r, "g_1"] & N \\ M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2 \arrow[u, "g_2"'] \end{tikzcd}$$

2. Denote by $\mathcal{K}^{am}_\mu$ the collection of models in $\mathcal{K}_\mu$ that are amalgamation bases.

De£nition. We say that $\mathcal{K}$ satis£es the *amalgamation property (AP)* iff every model $M \in \mathcal{K}$ is an amalgamation base.

TYPES

Proposition. $\sim$ is an equivalence relation on

$\{(M, N, \bar{a}) \in \mathcal{K}_\mu^3 \mid M, N$ are amalgamation bases$\}$.

De£nition

1. For $M \prec_{\mathcal{K}} N \in \mathcal{K}_\mu^{am}$ and $\mathbf{a} \in N$, we let

$$\mathtt{tp}(a/M, N) := (M, N, a)/\sim .$$

   This is the *type of a over M in N*.

2. For $M \in \mathcal{K}_\mu^{am}$, we let $S(M) :=$

$$\{\ \mathtt{tp}(a'/M, N)\ :\ M \prec_{\mathcal{K}} N \in \mathcal{K}_\mu^{am}, a \in N\}.$$

Density of Amalgamation Bases in the context of Shelah and Villaveces

Theorem. [Shelah-Villaveces]
Let μ be a cardinal with $LS(\mathcal{K}) \leq \mu < \lambda$. Then for every $M \in \mathcal{K}_\mu$, there exists $M' \in \mathcal{K}^{am}_\mu$ with $M \prec_{\mathcal{K}} M'$.

Theorem. [V]
Let μ be a cardinal with $LS(\mathcal{K}) \leq \mu < \lambda$. Fix $N_\lambda \in \mathcal{K}_\lambda$ Suppose $M \in \mathcal{K}_\mu$ and $\bar{a} \in {}^{\mu}|N_\lambda|$. If $M \prec_{\mathcal{K}} N_\lambda$, then there exists $M^{\bar{a}}$ an amalgamation base of cardinality μ such that $M \bigcup \bar{a} \subseteq M^{\bar{a}} \prec_{\mathcal{K}} N_\lambda$.

Saturated Models

De£nition. A model $N \in \mathcal{K}_\mu$ is *saturated* iff for every $M \prec_\mathcal{K} N$ with $M \in \mathcal{K}^{am}_{<\mu}$ and every type $p \in S(M)$, there exists a realization of p in N.

De£nition. An AEC, $\mathcal{K}$, is *μ-stable* iff for every $M \in \mathcal{K}^{am}_\mu$, we have that $|S(M)| \leq \mu$.

Proposition. If $\mathcal{K}$ is μ-stable and has the amalgamation property, then there exists a saturated model N of cardinality μ. Moreover, the saturated model of cardinality μ is unique up to isomorphism.

Remark. Without the amalgamation property, the existence of a saturated model is not known in general.

Rough Sketch of Attempts towards the Categoricity Conjecture

Show that:

1. For every μ there exists a saturated model of cardinality μ.
2. Limit models of a given cardinality are unique up to isomorphism.
3. A model is saturated iff it is a limit model.
4. If $M \in \mathcal{K}_\mu$ is not saturated, then there exists $N \in \mathcal{K}_\lambda$ which is not saturated.

LIMIT MODELS

De£nition. We say $N \in \mathcal{K}$ *is universal over* M iff for every $M' \in \mathcal{K}_{|M|}$ with $M \prec_{\mathcal{K}} M'$, there exists $f : M' \hookrightarrow N$ such that $f \restriction M = id_M$.

De£nition. Let μ be a cardinal and θ a limit ordinal with $\theta < \mu^+$. We say that N is a (μ, θ)-limit over M provided there exists $\langle N_i \mid i < \theta \rangle$ such that for every $i < \theta$,

1. $N_i \in \mathcal{K}^{am}_{\mu}$
2. N_{i+1} is universal over N_i
3. $N_0 = M$ and
4. $\bigcup_{i<\theta} N_i = N$.

EXISTENCE OF LIMIT MODELS

Theorem. [Shelah-Villaveces] Let $\mu < \lambda$. For every $M \in \mathcal{K}^{am}_\mu$, there exists $N \in \mathcal{K}_{\|M\|}$ with N universal over M.

Corollary. [Shelah-Villaveces] Let μ be a cardinal and θ a limit ordinal such that $\theta < \mu^+ \leq \lambda$. Then there exists a (μ, θ)-limit, $M \in \mathcal{K}$.

Theorem. [Shelah-Villaveces] If M is a (μ, θ)-limit for some $\theta < \mu^+ \leq \lambda$, then M is an amalgamation base.

Theorem. [V] Let $\mu < \lambda$. For every $M \in \mathcal{K}^{am}_\mu$, there exists $N \in \mathcal{K}_{\mu^+}$ such that N is μ^+-universal over M.

De£nition. We say $N \in \mathcal{K}$ *is* κ*-universal over* $M \in \mathcal{K}_\mu$ iff for every $M' \in \mathcal{K}_\kappa$ with $M \prec_{\mathcal{K}} M'$, there exists $f : M' \hookrightarrow N$ with $f \restriction M = id_M$.

UNIQUENESS OF LIMIT MODELS

Let $\mu < \lambda$ and θ a limit ordinal with $\theta < \mu^+$.

Conjecture. Let $\mu < \lambda$ and θ_1, θ_2 be limit ordinals with $\theta_1, \theta_2 < \mu^+$. If N_1 and N_2 are (μ, θ_1)- and (μ, θ_2)-limits over M, respectively, then $N_1 \cong N_2$.

Proposition. [Shelah-Villaveces] If N_1 and N_2 are (μ, θ)-limits over M, then there exists $f : N_1 \cong N_2$ with $f \restriction M = id_M$.

Proposition. [Shelah-Villaveces] If N_1 is a (μ, θ)-limit over M and N_2 is a $(\mu, \mathrm{cf}(\theta))$-limit over M, then $N_1 \cong N_2$.

Towards Uniqueness of Limit Models

Approach the Uniqueness Conjecture in stages:

1. Prove that if $N \in \mathcal{K}_\lambda$ is weakly model homogeneous, then any 2 limit models of the same cardinality are isomorphic.

2. Prove that $N \in \mathcal{K}_\lambda$ is weakly model homogeneous.

Model Homogeneity

De£nition. N is *model homogeneous* iff for every $M \prec_{\mathcal{K}} N$ with $M \in \mathcal{K}_{<\|N\|}$, we have that N is universal over M.

Remark. If $N \in \mathcal{K}_\lambda$ (where λ is the categoricity cardinal) is model homogeneous, then $\mathcal{K}$ has the amalgamation property below λ.

Proposition. [Shelah]
If $\mathcal{K}$ satis£es the amalgamation property , $M \in \mathcal{K}$ is saturated iff M is model homogeneous.

Remark. In particular, if $\mathcal{K}$ has the amalgamation property and is categorical in λ, then $N \in \mathcal{K}_\lambda$ is saturated and hence model homogeneous.

De£nition. N is *weakly model homogeneous (wmh)* iff for every $M \prec_{\mathcal{K}} N$ with $M \in \mathcal{K}^{am}_{<\|N\|}$, we have that N is universal over M.

In the context of Shelah-Villaveces assuming that $N \in \mathcal{K}_\lambda$ weakly model homogeneous:

1. [easy] $N \in \mathcal{K}_\lambda$ is saturated.

2. [V] If M is a (μ, θ)-limit for some $\theta < \mu^+ \leq \lambda$, then there exists an increasing and continuous sequence of ordinals $\langle \alpha_i \mid i < \theta \rangle$ such that
$$M \cong EM(I_{sup_{i<\theta}\{\alpha_i\}}, \Phi) \restriction L(\mathcal{K})$$
and $\langle EM(I_{\alpha_i}, \Phi) \restriction L(\mathcal{K}) \mid i < \theta \rangle$ witnesses that $EM(I_{sup_{i<\theta}\{\alpha_i\}}, \Phi) \restriction L(\mathcal{K})$ is a (μ, θ)-limit.

3. [V] (Uniqueness of limit models) Suppose θ_l is a limit ordinal with $\theta_l < \mu^+ \leq \lambda$ (for $l = 1, 2$). If N_l are (μ, θ_l)-limits for $l = 1, 2$, then there exists an isomorphism $f : N_1 \cong N_2$.

Conjecture. In the context of Shelah and Villaveces, the categoricity model is weakly model homogeneous.

De£nition. Θ_{μ^+} is said to hold if and only if for all $\{f_\eta \mid \eta \in {}^{\mu^+}2$ where $f_\eta : \mu^+ \to \mu^+\}$ and for every club $C \subseteq \mu^+$, there exists $\eta \neq \nu \in {}^{\mu^+}2$ and there exists a $\delta \in C$ such that

1. $\eta \restriction \delta = \nu \restriction \delta$,

2. $f_\eta \restriction \delta = f_\nu \restriction \delta$ and

3. $\eta[\delta] \neq \nu[\delta]$.

Remark Devlin and Shelah have shown that Θ_{μ^+} follows from $2^\mu < 2^{\mu^+}$.