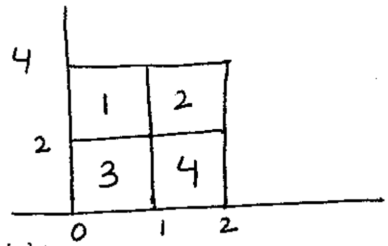
Math 21-259 Calculus in 3D
Homework 9 Solution
Spring 2011

1. **Solution:** Draw the rectangle $[0,2] \times [0,4]$ and partition it into four sub-rectangles. Label them starting from 1 to 4. Note, we are given that $f(x,y) = x + 2y^2$.



(a) Sample points = lower right corner.

Rectangle label	(x_i^*, y_i^*)	$f(x_i^*, y_i^*)$	Δx_i
1.	(1, 2)	9	2
2.	(2, 2)	10	2
3.	(1, 0)	1	2
4.	(2, 0)	2	2

The Estimated Volume = 9(2) + 10(2) + 1(2) + 2(2) = 44

(b) Sample points = Midpoints of each rectangle.

Rectangle label	(x_i^*, y_i^*)	$f(x_i^*, y_i^*)$	Δx_i
1.	(1/2, 3)	37/2	2
2.	(3/2, 3)	39/2	2
3.	(1/2, 1)	5/2	2
4.	(3/2,1)	7/2	2

The Estimated Volume = 37/2(2) + 39/2(2) + 5/2(2) + 7/2(2) = 88.

(c) Actual Volume = V = $\int_0^4 \int_0^2 (x + 2y^2)\,\text{dxdy}$

$$\begin{aligned} V &= \int_0^4 [\frac{x^2}{2} + 2xy^2]_{x=0}^{x=2}\,\text{dy} \\ &= \int_0^4 2 + 4y^2\,\text{dy} \\ &= \left[2y + 4\frac{y^3}{3}\right]_0^4 = \frac{280}{3}. \end{aligned}$$

Thus, we see that the estimate from the Midpoint in part(b) is much closer to the true vale than the estimate in part(a).

2. **Solution:** Let $I = \int_2^4 \int_{-1}^1 (x^2 + y^2)\,\mathrm{dydx}$. Consider,

$$\begin{aligned} I &= \int_2^4 [x^2y + \frac{y^3}{3}]_{y=-1}^{y=1}\,\mathrm{dx} \\ &= \int_2^4 (2x^2 + \frac{2}{3})\,\mathrm{dx} \\ &= \left[\frac{2}{3}x^3 + \frac{2}{3}x\right]_2^4 = \frac{116}{3}. \end{aligned}$$

3. **Solution:** Let $I = \int_0^1 \int_1^2 \frac{xe^x}{y}\, dydx$. Consider,

$$\begin{aligned} I &= \int_0^1 xe^x\,\mathrm{dx} \int_1^2 \frac{1}{\mathrm{y}}\,\mathrm{dy} \\ &= [xe^x - e^x]_0^1 [\ln y]_1^2 \qquad \text{[Integration by parts]} \\ &= [(e-e) - (0-1)](\ln 2 - \ln 1) = \ln 2. \end{aligned}$$

4. **Solution:** Let $I = \iint_R \frac{1+x^2}{1+y^2}\,\mathrm{dA}$, where $R = [0,1] \times [0,1]$. Consider,

$$\begin{aligned} I &= \int_0^1 (1+x^2)\,\mathrm{dx} \int_0^1 \frac{1}{1+\mathrm{y}^2}\,\mathrm{dy} \\ &= [x + \frac{x^3}{3}]_0^1\,[\arctan y]_0^1 \\ &= [1 + 1/3 - 0](\pi/4 - 0) = \pi/3. \end{aligned}$$

5. Here we need the volume of the solid lying under the surface $z = 1 + (x-1)^2 + 4y^2$ and above the rectangle $R = [0,3] \times [0,2]$ in the xy-plane. Thus,

$$\begin{aligned} V &= \int_0^3 \int_0^2 (1 + (x-1)^2 + 4y^2)\,\mathrm{dydx} \\ &= \int_0^3 [y + y(x-1)^2 + 4\frac{y^3}{3}]_{y=0}^{y=2}\,\mathrm{dx} \\ &= \int_0^3 [2 + 2(x-1)^2 + \frac{32}{3}]\,\mathrm{dx} \\ &= [2x + 2\frac{(x-1)^3}{3} + \frac{32}{3}x]_0^2 = 44. \end{aligned}$$

6. **Solution:** We need to find $I = \iint_D \frac{4y}{x^3+2}\,\mathrm{dA}$, where $D = \{(x, y)|1 \le x \le 2, 0 \le y \le 2x\}$. Consider,

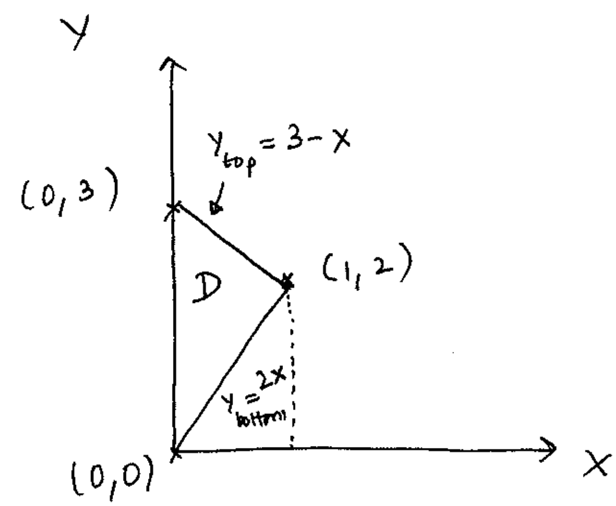
$$\begin{aligned} I &= \int_1^2 \int_0^{2x} \frac{4y}{x^3+2}\,\mathrm{dydx} \\ &= \int_1^2 [\frac{2y^2}{x^3+2}]_{y=0}^{y=2x}\,\mathrm{dx} \\ &= \int_1^2 [\frac{8x^2}{x^3+2} - 0]\,\mathrm{dx} \\ &= \int_{1^3+2}^{2^3+2} \frac{8}{3u}\,\mathrm{du} \qquad [\mathrm{Use\ u-sub,\ u = x^3+2}] \\ &= \frac{8}{3}[\ln(10) - \ln(3)] = \frac{8}{3}\ln\frac{10}{3}. \end{aligned}$$

7. **Solution:** We need to find $I = \iint_D e^{y^2}\,\mathrm{dA}$, where $D = \{(x, y)|0 \le y \le 1, 0 \le x \le y\}$. Consider,

$$\begin{aligned} I &= \int_0^1 \int_0^{y} e^{y^2}\,\mathrm{dxdy} \\ &= \int_0^1 [xe^{y^2}]_{x=0}^{x=y}\,\mathrm{dy} \\ &= \int_0^1 ye^{y^2}\,\mathrm{dy} \\ &= \int_{0^2}^{1^2} \frac{1}{2}e^u\,\mathrm{du} \qquad [\mathrm{Use\ u-sub, u = y^2}] \\ &= \frac{1}{2}[e^1 - e^0] = \frac{1}{2}(e-1). \end{aligned}$$

8. **Solution:** We need to find $I = \iint_D 2xy\,\mathrm{dA}$, where D is the triangular region with vertices (0, 0), (1, 2), and (0, 3).

Step 1. Sketch the region.



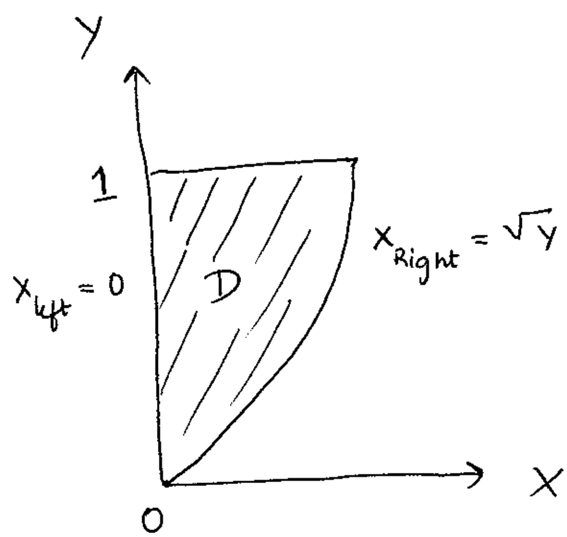
Step 2. Identify it as type I or type II and decide the order of integration. We set up the integral as follows:

$$\begin{aligned} I &= \int_0^1 \int_{2x}^{3-x} 2xy \, \mathrm{dydx} \\ &= \int_0^1 \left[xy^2\right]_{y=2x}^{y=3-x} \mathrm{dx} \\ &= \int_0^1 x(3-x)^2 - x(2x)^2 \, \mathrm{dx} \\ &= [-\frac{3}{4}x^4 - 2x^3 + \frac{9}{2}x^2]_0^1 \, \mathrm{dx} \\ &= -\frac{3}{4} - 2 + \frac{9}{2} = \frac{7}{4}. \end{aligned}$$

9. **Solution:** Here we need the volume of the solid lying under the surface $z = \sqrt{4-y^2}$(since in the first octant) and above the triangle with vertices (0, 0), (4, 2), and (0, 2) in the xy-plane(You should sketch the domain in the xy-plane!).
Thus,

$$\begin{aligned} V &= \int_0^2 \int_0^{2y} \sqrt{4-y^2} \, \mathrm{dxdy} \\ &= \int_0^2 \left[x\sqrt{4-y^2}\right]_{x=0}^{x=2y} \mathrm{dy} \\ &= \int_0^2 2y\sqrt{4-y^2} \, \mathrm{dy} \\ &= \int_{4-0^2}^{4-2^2} -\sqrt{u} \, \mathrm{du} \qquad [\mathrm{Use\ u-sub,\ u = 4 - y^2}] \\ &= [-\frac{2}{3}u^{3/2}]_4^0 = \frac{2}{3}4^{3/2} = \frac{16}{3}. \end{aligned}$$

10. **Solution:** To reverse the order of integration, it is important to sketch the given domain.



Note that the given integral as be written as follows:

$$
\begin{aligned}
I &= \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\,\mathrm{dxdy} \\
&= \int_0^1 [\frac{x^4}{4} \sin(y^3)]_{x=0}^{x=\sqrt{y}}\,\mathrm{dx} \\
&= \int_0^1 \frac{y^2}{4} \sin(y^3)\,\mathrm{dy} \\
&= \int_{0^3}^{1^3} \frac{1}{12} \sin u\,\mathrm{du} \qquad [\mathrm{Use\ u-sub,\ \ u = y^3}] \\
&= -\frac{1}{12}(\cos 1 - \cos 0) = \frac{1}{12}(1 - \cos 1).
\end{aligned}
$$