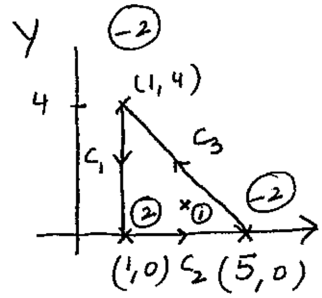
Math 21-259 Calculus in 3D
Homework 8 Solution
Spring 2011

1. **Solution:** We are given that $f(x,y) = e^{4y-x^2-y^2}$. To find critical points, set $f_x = -2xe^{4y-x^2-y^2}0$ and $f_y = (4-2y)e^{4y-x^2-y^2} = 0$. This implies that (0, 2) is the only critical point.

   Second Derivative test : $f_{xx}(x,y) = (4x^2-2)e^{4y-x^2-y^2}$, $f_{xy}(x,y) = 2x(4-2y)e^{4y-x^2-y^2}$, and $f_{yy}(x,y) = (4y^2-16y+14)e^{4y-x^2-y^2}$. Note: $A = f_{xx}(0,2) = -2e^4 < 0$, $B = f_{xy}(0,2) = 0$, $C = f_{yy}(0,2) = -2e^4, D = AC - B^2 = (-2e^4)(-2e^4) > 0$. This implies that (0, 2) is a point of local minimum and $f(0,2) = e^4$ is the local minimum value.

2. **Solution:** We are given that $f(x,y) = 3 + xy - x - 2y$.
   Step 1. Sketch the Region.



   Step 2. Find critical points in the interior of the domain.

$$\nabla f = < y-1, x-2 > = \mathbf{0} \Rightarrow y = 1, x = 2.$$

   Label it!
   Note that the point (2, 1) lies in the interior of the domain. Record the value of $f(2,1) = 1$.

   Step 3. Find the extreme values on the boundary of the domain.

   Parametrize C_1: $x(t) = 1,\ y(t) = 4-4t,\ 0 \le t \le 1$.
   Here $f_1(t) = f(x(t), y(t)) = 3 + (4-4t) - 1 - 2(4-4t)$.
   Note $f_1'(t) \neq 0$, No critical points!
   $f_1(0) = f(1,4) = -2,\ f_1(1) = f(1,0) = 2$.

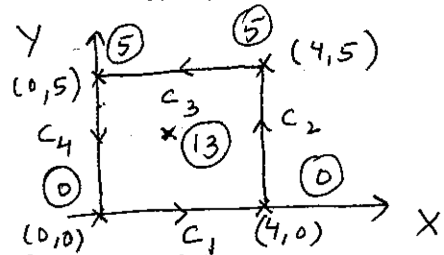
   Parametrize C_2: $x(t) = 1+4t,\ y(t) = 0,\ 0 \le t \le 1$.
   Here $f_2(t) = f(x(t), y(t)) = 3 - (1+4t)$.
   Note $f_2'(t) \neq 0$, No Critical Points!
   $f_2(0) = f(1,0) = 2,\ f_2(1) = f(5,0) = -2$.

   Parametrize C_3: $x(t) = 5-4t,\ y(t) = 4t,\ 0 \le t \le 1$.
   Here $f_3(t) = f(x(t), y(t)) = 3 + (5-4t)(4t) - (5-4t) - 8t$.
   Note $f_3'(t) = 4(5-4t) - 16t + 4 - 8 = 0 \ \Rightarrow 16 - 32t = 0 \ \Rightarrow t = 1/2$.
   $f_3(0) = f(5,0) = -2,\ f_3(1) = f(1,4) = -2,\ f_3(1/2) = f(3,2) = 2$.

   Absolute Maximum = 2 at (1, 0) and (3, 2)
   Absolute Minimum = -2 at (1, 4) and (5, 0).

3. **Solution:** We are given that $f(x, y) = 4x + 6y - x^2 - y^2$.

Step 1. Sketch the Region: Rectangle $= \{(x, y) : 0 \leq x \leq 4,\ 0 \leq y \leq 5\}$



Step 2. Find critical points in the interior of the domain.

$$\nabla f = < 4 - 2x, 6 - 2y > = \mathbf{0} \Rightarrow x = 2, y = 3.$$

Label it!

Note that the point (2, 3) lies in the interior of the domain. Record the value of $f(2, 3) = 13$.

Step 3. Find the extreme values on the boundary of the domain.

Parametrize C_1: $x(t) = 4t,\ y(t) = 0,\ 0 \leq t \leq 1$.
Here $f_1(t) = f(x(t), y(t)) = 16t - 16t^2$.
Note $f_1'(t) = 0 \Rightarrow t = 1/2$ - critical point.
$f_1(0) = f(0, 0) = 0,\ f_1(1/2) = f(2, 0) = 4,\ f_1(1) = f(4, 0) = 0$.

Parametrize C_2: $x(t) = 4,\ y(t) = 5t,\ 0 \leq t \leq 1$.
Here $f_2(t) = f(x(t), y(t)) = 30t - 25t^2$.
Note $f_2'(t) = 0 \Rightarrow t = 3/5$.
$f_2(0) = f(4, 0) = 0,\ f_2(1) = f(4, 5) = 5,\ f_2(3/5) = f(4, 3) = 9$.

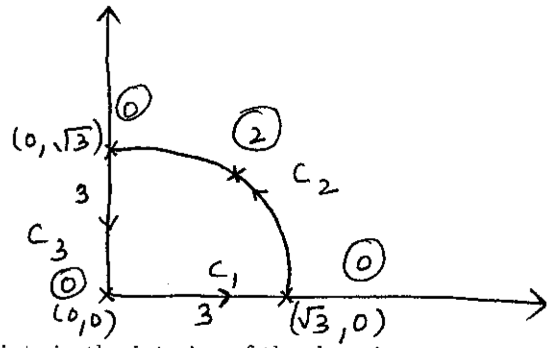
Parametrize C_3: $x(t) = 4 - 4t,\ y(t) = 5,\ 0 \leq t \leq 1$.
Here $f_3(t) = f(x(t), y(t)) = 4(4 - 4t) + 6(5) - (4 - 4t)^2 - 25$.
Note $f_3'(t) = -16 + 8(4 - 4t) = 0\ \Rightarrow t = 1/2$.
$f_3(0) = f(4, 5) = 5,\ f_3(1) = f(0, 5) = 5,\ f_3(1/2) = f(2, 5) = 9$.

Parametrize C_4: $x(t) = 0,\ y(t) = 5 - 5t,\ 0 \leq t \leq 1$.
Here $f_4(t) = f(x(t), y(t)) = 6(5 - 5t) - (5 - 5t)^2$.
Note $f_4'(t) = -30 - 2(5 - 5t)(-5) = 0\ \Rightarrow t = 2/5$.
$f_4(0) = f(0, 5) = 5,\ f_4(1) = f(0, 0) = 0,\ f_4(2/5) = f(0, 3) = 9$.

Absolute Maximum = 13 at (2, 3)
Absolute Minimum = 0 at (4, 0) and (0, 0).

4. **Solution:** We are given that $f(x, y) = xy^2$.

Step 1. Sketch the Region: Quarter Circle $= \{(x, y) : 0 \leq x,\ 0 \leq y, x^2 + y^2 \leq 3\}$



Step 2. Find critical points in the interior of the domain.

$$\nabla f =< y^2, 2xy >= \mathbf{0} \Rightarrow y = 0, x = 0.$$

Label it!

There are no critical points in the interior of the domain. Ignore it!

Step 3. Find the extreme values on the boundary of the domain.

On C_1**:** $x(t) = \sqrt{3}t,\ y(t) = 0,\ 0 \leq t \leq 1.$

Here,

$$f_1(t) = f(x(t), y(t)) = 0.$$

On C_2**:** Note that on the piece of circle, it is easier to use equation in rectangular coordinates instead of parametrizing. We know on C_2, $x^2 + y^2 = 3 \Rightarrow\ y = \sqrt{3 - x^2}$ since it is in the positive quadrant. We use this in the given function to find a function of one variable on the curve.

Here,

$$f_2(x) = f(onC_2) = x(3 - x^2),\ 0 \leq x \leq \sqrt{3}.$$

Note $f_2'(x) = 0 \Rightarrow 3 - 3x^2 = 0 \Rightarrow x = \pm 1$. Ignore $x = -1$!

$f_2(0) = f(0, \sqrt{3}) = 0,\ f_2(\sqrt{3}) = f(\sqrt{3}, 0) = 0,\ f_2(1) = f(1, \sqrt{2}) = 2.$

On C_3**:** $x(t) = 0,\ y(t) = \sqrt{3} - \sqrt{3}t,\ 0 \leq t \leq 1.$

Here $f_3(t) = f(x(t), y(t)) = 0.$

Absolute Maximum $= 2$ at $(1, \sqrt{2})$

Absolute Minimum $= 0$ at all points along C_1 and C_3.

5. **Solution:** To find the point on the plane $x - y + z = 4$ that is closest to the point (1, 2, 3), we minimize the distance function. Let d be the distance from a point (x, y, z) on the plane to the point 1, 2, 3). Then

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}.$$

Note that minimizing d is same as minimizing d^2 and $z = 4 - x + y$ using the equation of the plane. Thus we try to minimize the function

$$f(x,y) = d^2 = (x-1)^2 + (y-2)^2 + ((4-x+y)-3)^2.$$

Here $f_x(x,y) = 4x - 2y - 4 = 0$ and $f_y = 4y - 2x - 2 = 0$. To find the critical points, solve these equations simultaneously. One way to do that is multiply the first equation by 2 and add the new first and the second equation to get $6x - 10 = 0 \Rightarrow x = 5/3$. Thus, we get (5/3, 4/3) as the only critical point. This point must correspond to the minimum distance, so the point on the plane closest to (1, 2, 3) is (5/3, 4/3, 11/3).

6. **Solution:** Let x, y, z denote the length, width and the height of the rectangular box respectively. Note that the Surface area of the rectangular box is given by $S = 2(xy + yz + zx) = 64cm^2$. Note $z = \frac{32-xy}{x+y}$.

Maximize the Volume function, $f(x,y) = xy\frac{32-xy}{x+y}$. Then $f_x(x,y) = \frac{32y^2-2xy^3-x^2y^2}{(x+y)^2} = y^2\frac{32-2xy-y^2}{(x+y)^2}$ and $f_y(x,y) = x^2\frac{32-2xy-x^2}{(x+y)^2}$. Set $f_x = 0$ and $f_y = 0$ which implies

$$32 - 2xy - x^2 = 0 \text{ and } 32 - 2\text{xy} - \text{y}^2 = 0.$$

You may now use your best way to solve these equations simultaneously. One way to do so is by setting $32 - 2xy - x^2 = 32 - 2xy - y^2 = 0$. This implies that $x^2 = y^2 \Rightarrow x = y$ since both x and y are positive. By Substituting $x = y$ in any of the two equations above, we get $32 - 2x^2 - x^2 = 0 \Rightarrow x^2 = 32/3$. Thus, $x = y = 4\sqrt{\frac{2}{3}}$ and $z = 4\sqrt{\frac{2}{3}}$. Thus the box is a cube with edges $4\sqrt{\frac{2}{3}}$.

7. **Solution:** Maximize/Minimize $f(x,y,z) = 8x - 4z$ subject to the constraint $x^2 + 10y^2 + z^2 = 5$.
Lagrange Condition: $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$.
Note $\nabla f(x,y,z) = < 8, 0, -4 >$ and $\nabla g(x,y,z) = < 2x, 20y, 2z >$. Thus the system of equations that we need to solve is given the following equations.

$$\begin{aligned} 2\lambda x &= 8 \\ 20\lambda y &= 0 \\ 2\lambda z &= -4 \\ x^2 + 10y^2 + z^2 &= 5. \end{aligned}$$

In this particular system, our strategy is solve for x, y, z in terms of λ and plug them into the side condition to get the value of λ and consequently the values of x, y, z.

Note $\lambda \neq 0$ which implies $x = \frac{4}{\lambda}, y = 0, z = -\frac{2}{\lambda}$ and hence $(\frac{4}{\lambda})^2 + (\frac{-2}{\lambda})^2 = 5 \Rightarrow \lambda = \pm 2$, so f has extreme values at the points $(2, 0, -1)$ and $(-2, 0, 1)$. The maximum value of f on $x^2 + 10y^2 + z^2 = 5$ is $f(2, 0, -1) = 20$, and the minimum is $f(-2, 0, 1) = -20$.

8. **Solution:** Maximize/ Minimize $f(x, y, z) = x^2y^2z^2$ subject to the constraint $x^2+y^2+z^2 = 1$.
Lagrange Condition: $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.
Note $\nabla f(x, y, z) =< 2xy^2z^2, 2x^2yz^2, 2x^2y^2z >$ and $\nabla g(x, y, z) =< 2x, 2y, 2z >$. Thus the system of equations that we need to solve is given the following set of equations.

$$\begin{aligned} 2\lambda x &= 2xy^2z^2 \\ 2\lambda y &= 2x^2yz^2 \\ 2\lambda z &= 2x^2y^2z \\ x^2 + y^2 + z^2 &= 1. \end{aligned}$$

In this particular system, our strategy is to solve for λ from the first three equations and plug them in the side condition to get the value of x, y, z.

Note we get two set of equations

$$\lambda \neq 0 \text{ and } \lambda = x^2y^2 = y^2z^2 = z^2x^2. \quad (1)$$
$$\lambda = 0 \text{ and one or two(but not three) of the coordinates are 0.} \quad (2)$$

Using equation (1) in the side condition, we get $x^2 = y^2 = z^2 = \frac{1}{3}$. Thus, the minimum value of f on the sphere occurs on case(2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, and $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

9. **Solution:** Maximize/Minimize $f(x, y, z) = x^4 + y^4 + z^4$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 1$.
Lagrange Condition: $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.
Note $\nabla f(x, y, z) =< 4x^3, 4y^3, 4z^3 >$ and $\nabla g(x, y, z) =< 2x, 2y, 2z >$. Thus the system of equations that we need to solve is given the following set of equations.

$$\begin{aligned} 2\lambda x &= 4x^3 \\ 2\lambda y &= 4y^3 \\ 2\lambda z &= 4z^3 \\ x^2 + y^2 + z^2 &= 1. \end{aligned}$$

Note $2\lambda x = 4x^3$ implies that either $x = 0$ or $\lambda = 2x^2$. Similarly other equations implies that either $y = 0$ or $\lambda = 2y^2$ or either $z = 0$ or $\lambda = 2z^2$. From this the following eight cases arise:

Case 1. $x = 0, y = 0, z = 0$: Note that this case is not possible since (0, 0, 0) does not satisfy the side condition.

Case 2. $x = 0, y = 0, \lambda = 2z^2$: By using these values in the side condition, we get that $z = \pm 1$. Thus there are two critical points which are $(0, 0, \pm 1)$ and $f(0, 0, \pm 1) = 1$.

Case 3. $x = 0, \lambda = 2y^2, z = 0$: By using these values in the side condition, we get that $y = \pm 1$. Thus there are two critical points which are $(0, \pm 1, 0)$ and $f(0, \pm 1, 0) = 1$.

Case 4. $\lambda = 2x^2, y = 0, z = 0$: By using these values in the side condition, we get that $x = \pm 1$. Thus there are two critical points which are $(\pm 1, 0, 0)$ and $f(\pm 1, 0, 0) = 1$.

Case 5. $x = 0, \lambda = 2y^2, \lambda = 2z^2$: By using these values in the side condition, we get that $y = \pm\frac{1}{\sqrt{2}}$ and $z = \pm\frac{1}{\sqrt{2}}$. Thus there are four critical points which are $(0, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(0, \pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and the value of the function at these critical points is $\frac{1}{2}$.

Case 6. $\lambda = 2x^2, y = 0, \lambda = 2z^2$: Same idea as Case(5). The value of the function at the critical points obtained in this case is also equal to $1/2$.

Case 7. $\lambda = 2x^2, \lambda = 2y^2, z = 0$: Same idea as Case(5). The value of the function at the critical points obtained in this case is also equal to $1/2$.

Case 8. $\lambda = 2x^2, \lambda = 2y^2, \lambda = 2z^2$: By using these values in the side condition, we get that $x = \pm\frac{1}{\sqrt{3}}, y = \pm\frac{1}{\sqrt{3}}$, and $z = \pm\frac{1}{\sqrt{3}}$. The value of the function at these critical points is $1/3$.

The final conclusion from all of the above cases is that the maximum value of the function is 1 and minimum value is $1/3$.

10. **Solution:** Maximize $f(x, y, z) = xyz$ subject to the constraint $xy + yz + zx = 32$.
Lagrange Condition: $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.
Note $\nabla f(x, y, z) = \langle yz, xz, xy \rangle$ and $\nabla g(x, y, z) = \langle y + z, x + z, x + y \rangle$. Thus the system of equations that we need to solve is given the following set of equations.

$$\begin{aligned} \lambda(y + z) &= yz & (3) \\ \lambda(x + z) &= xz & (4) \\ \lambda(x + y) &= xy & (5) \\ xy + yz + zx &= 32. & (6) \end{aligned}$$

Note, if we subtract (4) from (3) we get that either $x = y$ or $\lambda = z$. If $\lambda = z$ then equation (3) implies that z $= 0$ which is not possible z being a height of the rectangular box. Thus $x = y$. Similarly by subtracting (5) from (4), we get that either $y = z$ or $\lambda = x$. Again due to the same reasoning we see that $\lambda \neq x$ which implies $y = z$. Using the information derived above $x = y = z$ in the side condition we get that $x^2 = y^2 = z^2 = 32/3$. This implies that we only have one critical point, that is, $(4\sqrt{\frac{2}{3}}, 4\sqrt{\frac{2}{3}}, 4\sqrt{\frac{2}{3}})$ and the maximum volume is $\frac{128}{3}\sqrt{\frac{2}{3}}$.