Math 21-259 Calculus in 3D
Homework 7 Solution
Spring 2011

1. **Solution:** We need to find $\frac{\mathrm{dz}}{\mathrm{dt}}$ given that $z = x\ln(x+2y),\ x = \sin t,\ y = \cos t$.
Chain Rule: $\frac{\mathrm{dz}}{\mathrm{dt}} = \frac{\partial z}{\partial x}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\partial z}{\partial y}\frac{\mathrm{dy}}{\mathrm{dt}}$.
By Substituting we get,

$$\frac{\mathrm{dz}}{\mathrm{dt}} = \underbrace{[x.\frac{1}{x+2y} + 1.\ln(x+2y)]}_{\frac{\partial z}{\partial x}}\underbrace{\cos t}_{\frac{\mathrm{dx}}{\mathrm{dt}}} + \underbrace{[x.\frac{1}{x+2y}(2)]}_{\frac{\partial z}{\partial y}}\underbrace{(-\sin t)}_{\frac{\mathrm{dy}}{\mathrm{dt}}}.$$

2. **Solution:** We need to find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ given that $z = x/y,\ x = se^t,\ y = 1+se^{-t}$.
Chain Rule:

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} and\\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.\end{aligned}$$

By Substituting we get,

$$\begin{aligned}\frac{\partial z}{\partial s} &= \underbrace{\frac{1}{y}}_{\frac{\partial z}{\partial x}}\underbrace{(e^t)}_{\frac{\partial x}{\partial s}} + \underbrace{(-\frac{x}{y^2})}_{\frac{\partial z}{\partial y}}\underbrace{(e^{-t})}_{\frac{\partial y}{\partial s}} = \frac{e^t}{y} - \frac{xe^{-t}}{y^2}.\\ \frac{\partial z}{\partial t} &= \underbrace{\frac{1}{y}}_{\frac{\partial z}{\partial x}}\underbrace{(se^t)}_{\frac{\partial x}{\partial t}} + \underbrace{(-\frac{x}{y^2})}_{\frac{\partial z}{\partial y}}\underbrace{(-se^{-t})}_{\frac{\partial y}{\partial t}} = \frac{se^t}{y} + \frac{xse^{-t}}{y^2}.\end{aligned}$$

3. **Solution:** We are given that $y^5 + x^2y^3 = 1 + ye^{x^2}$, so we let $F(x,y) = y^5 + x^2y^3 - 1 - ye^{x^2} = 0$.
Then

$$\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{F_x}{F_y} = -\frac{2xy^3 - 2xye^{x^2}}{5y^4 + 3x^2y^2 - e^{x^2}} = \frac{2xye^{x^2} - 2xy^3}{5y^4 + 3x^2y^2 - e^{x^2}}.$$

4. **Solution:** We are given that $yz = \ln(x+z)$, so we let $F(x,y) = yz - \ln(x+z) = 0$. Then

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = \frac{-\frac{1}{x+z}(1)}{y - \frac{1}{x+z}(1)} = \frac{1}{y(x+z)-1} \quad \text{and}\\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{z}{y - \frac{1}{x+z}} = \frac{z(x+z)}{y(x+z)-1}.\end{aligned}$$

5. **Solution:** We are given that $f(x,y,z) = (x+yz)^{1/2}$, $P(1,3,1)$, and $\hat{\mathbf{u}} =< \frac{2}{7}, \frac{3}{7}, \frac{6}{7} >$.

(a) The gradient of f is given by

$$\begin{aligned}
\nabla f(x,y,z) &= < f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) > \\
&= < \frac{1}{2}(x+yz)^{-1/2}(1), \frac{1}{2}(x+yz)^{-1/2}(z), \frac{1}{2}(x+yz)^{-1/2}(y) > \\
&= < \frac{1}{(2\sqrt{x+yz})}, \frac{z}{(2\sqrt{x+yz})}, \frac{y}{(2\sqrt{x+yz})} > .
\end{aligned}$$

(b) The gradient of f at P is given by $\nabla f(1,3,1) =< \frac{1}{4}, \frac{1}{4}, \frac{3}{4} >$.

(c) The directional derivative of f in the direction of $\hat{\mathbf{u}}$ is given by

$$D_{\hat{\mathbf{u}}} f(1,3,1) = \nabla f(1,3,1).\hat{\mathbf{u}} =< \frac{1}{4}, \frac{1}{4}, \frac{3}{4} > . < \frac{2}{7}, \frac{3}{7}, \frac{6}{7} >= \frac{2}{28} + \frac{3}{28} + \frac{18}{28} = \frac{23}{28}.$$

6. **Solution:** We are given that $f(x,y) = \ln(x^2+y^2)$, $P(2,1)$, and $\hat{\mathbf{v}} = \frac{1}{\sqrt{2^2+(-1)^2}} < -1, 2 >$.

The directional derivative of f in the direction of the unit vector $\hat{\mathbf{v}}$ is given by

$$\begin{aligned}
D_{\hat{\mathbf{v}}} f(2,1) &= \nabla f(2,1).\hat{\mathbf{v}} \\
&= < \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} > |_{(x=2,y=1)}. < \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} > \\
&= < \frac{4}{5}, \frac{2}{5} > . < \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} >= 0.
\end{aligned}$$

7. **Solution:** We are given that $f(x,y,z) = x^2+y^2+z^2$ and $P(2,1,3)$
The unit vector in the direction of **PO** is given by $\hat{\mathbf{u}} = \frac{1}{\sqrt{14}} < -2, -1, -3 >$ and the gradient of f at P is given by $\nabla f(2,1,3) =< 2x, 2y, 2z > |_{(2,1,3)} =< 4, 2, 6 >$. Thus, the directional derivative of f in the direction of $\hat{\mathbf{u}}$, $D_{\hat{\mathbf{u}}} f(2,1,3) =< 4, 2, 6 > .\frac{1}{\sqrt{14}} < -2, -1, -3 >= -2\sqrt{14}.$

8. **Solution:** If $T(x,y,z) = 200e^{-x^2-3y^2-9z^2}$, then $\nabla T(x,y,x) = -400e^{-x^2-3y^2-9z^2} < x, 3y, 9z >$.

(a) Here, we are looking for the directional derivative of T at P(2, -1, 2) in the direction towards the point Q(3, -3, 3), that is, $\hat{\mathbf{u}} = \frac{\mathbf{PQ}}{|\mathbf{PQ}|} = \frac{1}{\sqrt{6}} < 1, -2, 1 >$. Thus,

$$\begin{aligned}
D_{\hat{\mathbf{u}}} T(2,-1,2) &= \nabla T(2,-1,2).\hat{\mathbf{u}} \\
&= -400e^{-43} < 2, -3, 18 > .\frac{1}{\sqrt{6}} < 1, -2, 1 >= \frac{5200\sqrt{6}}{3e^{43}}^{\circ} C/m.
\end{aligned}$$

(b) $\nabla T(2,-1,2) = 400e^{-43} < -2, 3, -18 >$ or equivalently $< -2, 3, -18 >$.

(c) $|\nabla T(x,y,z)| = |-400e^{-x^2-3y^2-9z^2} < x, 3y, 9z > | = -400e^{-x^2-3y^2-9z^2}\sqrt{x^2+9y^2+8z^2}^{\circ}C/m$ is the maximum rate of increase. At (2, -1, 2) the maximum rate of increase is $400e^{-43}\sqrt{337}^{\circ}C/m$.

9. **Solution:** Let $F(x,y,z) = x - z - 4\arctan(yz)$. Then $x - z = 4\arctan(yz)$ is the level surface $F(x,y,z) = 0$ and $\nabla F(x,y,z) = \left\langle 1, -\frac{4z}{1+y^2z^2}, -1 - \frac{4y}{1+y^2z^2} \right\rangle$.

(a) To find the equation of the tangent plane to the given surface at $P(1+\pi, 1, 1)$(given point), we need a point and a normal vector at that point.
Point $= P(1+\pi, 1, 1)$
Normal Vector at P $= \nabla F(1+\pi, 1, 1) =< 1, -2, -3 >$

Thus the equation is given by

$$1(x - (1+\pi)) - 2(y-1) - 3(z-1) = 0 \Leftrightarrow x - 2y - 3z = -4 + \pi.$$

(b) To find the equation of the normal line to the given surface at the given point $P(1+\pi, 1, 1)$, we need a point and a direction vector at that point.
Point $= P(1+\pi, 1, 1)$
Direction vector at P $= \nabla F(1+\pi, 11) =< 1, -2, -3 >$

Thus the equation is given by

$$x - (1+\pi) = \frac{y-1}{-5} = \frac{z-1}{-3}.$$

10. **Solution:** It is same as asking for the points at which the normal vector of the given ellipsoid is parallel to the normal vector of the given plane.

Normal vector of the given plane at any point (x, y, z)=$< 3, -1, 3 >$.
Normal vector of the ellipsoid at any point (x, y, z)=$< 2x, -2y, 6z >$.

The question reduces to finding (x,y,z) that satisfies the equation of the ellipsoid so that $< 2x, -2y, 6z >= k < 3, -1, 3 >$ for some constant k. Thus,

$$x = \frac{3}{2}k,\ y = \frac{-1}{-2}k,\ z = \frac{3}{6}k.$$

Substitute these values of x, y, z in the equation of the ellipsoid to get the value of k and hence the values of x, y, z. Consider,

$$\left(\frac{3}{2}k\right)^2 - \left(\frac{-1}{-2}k\right)^2 + 2\left(\frac{3}{6}k\right)^2 = 1 \Longleftrightarrow k = \pm\frac{\sqrt{2}}{5}.$$

Thus, we get two such points $\left(\pm\frac{3\sqrt{2}}{5}, \mp\frac{1}{5\sqrt{2}}, \pm\frac{\sqrt{2}}{5}\right)$.