Math 21-259 Calculus in 3D
Homework 15 Solution
Spring 2011

1. **Solution:** (13.6 P 2) [Courtesy: Tim Carson]

   For any given v, the curve is an ellipse in the $z = v$ plane. Also, $0 \leq v \leq 2$. Therefore the surface is an elliptical cylinder between $z = 0$ and $z = 2$.

2. **Solution:** (13.6 P 20) [Courtesy: Tim Carson]

   I will use spherical coordinates. I'll use u to represent θ and v to represent ϕ.

   $x^2 + y^2 + z^2 = 16$, so the radius is constantly 4.

   $z = \rho \cos v = 4 \cos v \leq 2$, so $\cos v \leq \frac{1}{2}$.

   $-2 \leq z = 4 \cos v$, so $\cos v \geq \frac{-1}{2}$.

   Therefore $\frac{\pi}{3} \leq v \leq \frac{2\pi}{3}$.

   There are no restrictions on x and y besides $x^2 + y^2 + z^2 = 16$, so $0 \leq u \leq 2\pi$.

   So the parameterization is

   $$r(u, v) =< 4 \sin v \cos u, 4 \sin v \sin u, 4 \cos v >$$

   where

   $$0 \leq u \leq 2\pi \qquad \frac{\pi}{3} \leq v \leq \frac{2\pi}{3}$$

   Another technique would be to use cylindrical coordinates, in which case one might get:

   $$< \sqrt{16 - z^2} \cos \theta, \sqrt{16 - z^2} \sin \theta, z >$$

   where

   $$0 \leq \theta \leq 2\pi \qquad -2 \leq z \leq 2$$

3. **Solution:** (13.6 P 34) [Courtesy: Tim Carson]

Write the plane as $z = 10 - 2x - 5y$. Then use remark 9 on page 772,

$$\begin{aligned} A(S) &= \int\int_R \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} dA \\ &= \int\int_R \sqrt{1 + 4 + 25}\, dA \end{aligned}$$

Use polar coordinates to evaluate this.

$$\int_0^{2\pi} \int_0^3 \sqrt{30} r\, dr d\theta = 2\pi\sqrt{30}\frac{9}{2}$$

4. **Solution:** (13.6 P 38) [Courtesy: Tim Carson]

Use remark 9 on page 772.

$$\begin{aligned} A(S) &= \int\int_R \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}\, dA \\ &= \int\int_R \sqrt{1 + 9 + 16y^2}\, dA \\ &= \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx dy \\ &= 2\int_0^1 y\sqrt{10 + 16y^2}\, dy \\ &= 2\left[\frac{1}{32}\frac{2}{3}(10 + 16y^2)^{3/2}\right]_0^1 \\ &= \frac{1}{24}(26^{3/2} - 10^{3/2}) \end{aligned}$$

5. **Solution:** (13.6 P 39) [Courtesy: Tim Carson]

Use remark 9 on page 772. Use polar coordinates to evaluate the integral.

$$\begin{aligned} \int\int_A \sqrt{1 + 4x^2 + 4y^2}\, dA &= \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} r\, dr d\theta \\ &= 2\pi\left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_1^2 \\ &= \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \end{aligned}$$

6. **Solution:** (13.7 P 6) [Courtesy: Jing Liu]
Note that $\mathbf{r}_u = \cos v\mathbf{i} + \sin v\mathbf{j}$, and $\mathbf{r}_v = -u\sin v\mathbf{i} + u\cos v\mathbf{j} + \mathbf{k}$. This implies $\mathbf{r}_u \times \mathbf{r}_v = \sin v\mathbf{i} - \cos v\mathbf{j} + u\mathbf{k}$. Therefore, $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1+u^2}$, so $\int\int_S \sqrt{1+x^2+y^2}\,dS = \int_0^\pi \int_0^1 \sqrt{1+u^2}\sqrt{1+u^2}\,du\,dv = \frac{4}{3}\pi$.

7. **Solution:**(13.7 P 8) [Courtesy: Jing Liu]

S is the region in the plane $2x+y+z=2$ or $z = 2-2x-y$ over $D = \{(x.y)\,|\,0 \le x \le 1, 0 \le y \le 2-2x\}$. Thus,

$$\begin{aligned}
\int\int_S xy\,dS &= \int\int_D xy\sqrt{(-2)^2+(-1)^2+1}\,dA \\
&= \sqrt{6}\int_0^1\int_0^{2-2x} xy\,dydx \\
&= \sqrt{6}\int_0^1 \left[\frac{1}{2}xy^2\right]_{y=0}^{y=2-2x} dx \\
&= \frac{\sqrt{6}}{2}\int_0^1 (4x-8x^2+4x^3)\,dx \\
&= \frac{\sqrt{6}}{2}(2-\frac{8}{3}+1) \\
&= \frac{\sqrt{6}}{6}
\end{aligned}$$

8. **Solution:**(13.7 P 9) [Courtesy: Jing Liu]
S is the part of the plane $z = 1-x-y$ over the region $D = \{(x,y)\,|\,0 \le x \le 1, 0 \le y \le 1-x\}$. Thus

$$\begin{aligned}
\int\int_S yz\,dS &= \int\int_D y(1-x-y)\sqrt{(-1)^2+(-1)^2+1}\,\mathrm{d}A \\
&= \sqrt{3}\int_0^1\int_0^{1-x}(y-xy-y^2)\,\mathrm{d}y\,\mathrm{d}x \\
&= \sqrt{3}\int_0^1\left[\frac{1}{2}y^2-\frac{1}{2}xy^2-\frac{1}{3}y^3\right]_{y=0}^{y=1-x} dx \\
&= \sqrt{3}\int_0^1 \frac{1}{6}(1-x)^3\,dx \\
&= -\frac{\sqrt{3}}{24}(1-x^4)\Big]_0^1 = \frac{\sqrt{3}}{24}.
\end{aligned}$$

9. **Solution:**(13.7 P 16) [Courtesy: Jing Liu]
Using cylindrical coordinates,

$$\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k},\ \ 0 \le \phi \le \frac{\pi}{4}, 0 \le \theta \le 2\pi,$$

and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$. (see Example 13.6.9). Then

$$\int\int_S xyz\,dS = \int_0^{2\pi}\int_0^{\pi/4}(\sin^3\phi\cos\phi\cos\theta\sin\theta)\,d\phi d\theta = 0,$$

since $\int_0^{2\pi} \cos\theta \sin\theta \, d\theta = 0$.

10. **Solution:**(13.7 P 23) [Courtesy: Jing Liu]
$\mathbf{F}(x,y,z) = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$, $z = g(x,y) = \sqrt{4-x^2-y^2}$ and D is the quarter disk

$$\{(x,y) \,|\, 0 \le x \le 2,\, 0 \le y \le \sqrt{4-x^2}\})$$

S has downward orientation, so by Formula 10,

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot d\mathbf{S} &= -\int\int_D [-\frac{x}{2}(4-x^2-y^2)^{-1/2}(-2x) - (-z)\cdot\frac{1}{2}(4-x^2-y^2)^{-1/2}(-2y) + y]\, dA \\
&= -\int\int_D \left(\frac{x^2}{\sqrt{4-x^2-y^2}} - \sqrt{4-x^2-y^2}\frac{y}{\sqrt{4-x^2-y^2}} + y\right) dA \\
&= -\int\int_D x^2(4-(x^2+y^2))^{-1/2}\, dA \\
&= -\int_0^{\pi/2}\int_0^2 (r\cos\theta)^2(4-r^2)^{-1/2} r\, dr\, d\theta \\
&= -\int_0^{\pi/2} \cos^2\theta\, d\theta \cdot \int_0^2 r^3(4-r^2)^{-1/2}\, dr \\
&= -\int_0^{\pi/2} (\frac{1}{2} + \frac{1}{2}\cos 2\theta)\, d\theta \int_4^0 -\frac{1}{2}(4-u)u^{-1/2}\, du \\
&= -\Big[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\Big]_0^{\pi/2}\big(-\frac{1}{2}\big)\Big[8\sqrt{u} - \frac{2}{3}u^{3/2}\Big]_4^0 \\
&= -\frac{\pi}{4}\big(-\frac{1}{2}\big)\big(-16 + \frac{16}{3}\big) \\
&= \frac{4}{3}\pi.
\end{aligned}$$

11. **Solution:**(13.7 P 25) [Courtesy: Paolo Piovano]

Let S_1 be the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and S_2 the disk $x^2 + z^2 \le 1$, $y = 1$. Since S is a closed surface, we use the outward orientation.

On S_1: $\mathbf{F}(\mathbf{r}(x,z)) = (x^2+z^2)\,\mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$, since the $\mathbf{j}$-component must be negative on S_1. Then,

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} &= \iint_{x^2+z^2\le 1} [-(x^2+z^2) - 2z^2]\, dA = -\int_0^{2\pi}\int_0^1 (r^2 + 2r^2\cos^2\theta) r\, dr\, d\theta \\
&= -\int_0^{2\pi} \frac{1}{4}(1 + 2\cos^2\theta)\, d\theta = -(\frac{\pi}{2} + \frac{\pi}{2}) = -\pi\,.
\end{aligned}$$

On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = \mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$. Then,

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} 1\, dA = \pi\,.$$

Thus,

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = -\pi + \pi = 0\,.$$

12. **Solution:**(13.7 P 26) [Courtesy: Paolo Piovano]

Here S consists of four surfaces: S_1, the triangular face with vertices $(1,0,0)$,$(0,1,0)$, and $(0,0,1)$; S_2, the face of the tetrahedron in the xz-plane; S_3, the face in the xz-plane; and S_4, the face in the yz-plane.

On S_1: the face is the portion of the plane $z = 1 - x - y$ for $0 \le x \le 1$, $0 \le y \le 1 - x$ with upward orientation, so

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{1-x} [-y(-1) - (z-y)(-1) + x]\, dy\, dx = \int_0^1 \int_0^{1-x} (z+x)\, dy\, dx \\ &= \int_0^1 \int_0^{1-x} (1-y)\, dy\, dx = \int_0^1 \left[y - \frac{y^2}{2} \right]_{y=0}^{y=1-x} dx = \frac{1}{2} \int_0^1 (1-x^2)\, dx \\ &= \frac{1}{2} \left[x - \frac{x^3}{3} \right]_0^1 = \frac{1}{3}\,.\end{aligned}$$

On S_2: the surface is $z = 0$ with downward orientation, so

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-x} (-x)\, dy\, dx = -\int_0^1 x(1-x)\, dx = -\left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = -\frac{1}{6}\,.$$

On S_3: the surface is $y = 0$ for $0 \le x \le 1$, $0 \le z \le 1 - x$ oriented in the negative y-direction. Regarding x and z as parameters, we have $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ and

$$\begin{aligned}\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{1-x} -(z-y)\, dz\, dx = -\int_0^1 \int_0^{1-x} z\, dz\, dx = -\int_0^1 \left[\frac{z^2}{2} \right]_{z=0}^{z=1-x} dx \\ &= -\frac{1}{2} \int_0^1 (1-x)^2\, dx = \frac{1}{6} \left[(1-x)^3 \right]_0^1 = -\frac{1}{6}\,.\end{aligned}$$

On S_4: the surface is $x = 0$ for $0 \le y \le 1$, $0 \le z \le 1 - y$ oriented in the negative x-direction. Regarding y and z as parameters, we have $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$, so we use $-\mathbf{r}_y \times \mathbf{r}_z = -\mathbf{i}$ and

$$\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-y} (-y)\, dz\, dy = -\int_0^1 y(1-y)\, dy = -\left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = -\frac{1}{6}\,.$$

Hence,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} = -\frac{1}{6}\,.$$

13. **Solution:**(13.8 P 1) [Courtesy: Paolo Piovano]

The boundary curve C is the circle $x^2 + y^2 = 4$, $z = 0$ oriented in the counterclockwise direction. The vector equation is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$ and

$$\mathbf{F}(\mathbf{r}(t)) = (2\cos t)^2 e^{(2\sin t)(0)}\,\mathbf{i} + (2\sin t)^2 e^{(2\cos t)(0)}\,\mathbf{j} + (0)^2 e^{(2\cos t)(2\sin t)}\,\mathbf{k} = 4\cos^2 t\,\mathbf{i} + 4\sin^2 t\,\mathbf{j}\,.$$

Then, by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-8\cos^2 t\sin t + 8\sin^2 t\cos t)\,dt$$
$$= 8\left[\frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t\right]_0^{2\pi} = 0\,.$$

14. **Solution:**(13.8 P 2) [Courtesy: Paolo Piovano]

The plane $z = 5$ intersects the paraboloid $z = 9 - x^2 - y^2$ in the circle $x^2 + y^2 = 4$, $z = 5$. This boundary curve C is oriented in the counterclockwise direction, so the vector equation is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 5\,\mathbf{k}$, $0 \le t \le 2\pi$. Then, $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 10\sin t\,\mathbf{i} + 10\cos t\,\mathbf{j} + 4\cos t\sin t\,\mathbf{k}$ and by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-20\sin^2 t + 20\cos^2 t)\,dt$$
$$= 20\int_0^{2\pi} \cos 2t\,dt = 0\,.$$

15. **Solution:**(13.8 P 6) [Courtesy: Paolo Piovano]
We have $\operatorname{curl}\mathbf{F} = e^x\,\mathbf{k}$ and S is the portion of the plane $2x + y + 2z = 2$ over $D = \{(x, y)\,|\,0 \le x \le 1, 0 \le y \le 2 - 2x\}$. Orienting S upward and using Equation 13.7.10 with $z = g(x, y) = 1 - x - \frac{y}{2}$ we obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D (0 + 0 + e^x)dA = \int_0^1 \int_0^{2-2x} e^x\,dy\,dx$$
$$= \int_0^1 (2 - 2x)e^x\,dx = [(2 - 2x)e^x + 2e^x]_0^1 = 2e - 4\,.$$

16. **Solution:**(13.8 P 8) Note that the surface is the part of the plane $z = 5 - x = g(x, y)$ that lies above the disc $x^2 + y^2 \le 9$ and $\operatorname{curl}\mathbf{F} = \mathbf{i} - x\mathbf{k}$. According to Stokes' Theorem, we get

$$\begin{aligned}
\oint_C \mathbf{F}.\mathrm{d}\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}.\mathrm{d}\mathbf{S} \\
&= \iint_{x^2+y^2\le 9} -g_x + (-x)\,\mathrm{d}A \\
&= \iint_{x^2+y^2\le 9} (1 - x)\,\mathrm{d}A \\
&= \int_0^{2\pi} \int_0^3 (1 - r\cos\theta)r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} (\frac{9}{2} - 9\cos\theta)\,\mathrm{d}\theta \\
&= \left[\frac{9}{2}\theta - 9\sin\theta\right]_0^{2\pi} = 9\pi.
\end{aligned}$$

17. **Solution:**(13.8 P 12) We need to show that

$$\oint_C \mathbf{F}.\mathrm{d}\mathbf{S} = \iint_S \operatorname{curl}(\mathbf{F}).\mathrm{d}\mathbf{S}.$$

L.H.S: The plane intersects the coordinate axes at $x = 1, y = z = 2$ so the boundary curve C consists of the following three line segments:
$\mathbf{C_1}$: is the line segment joining (1, 0, 0) and (0, 2, 0). Thus, it can parametrized as

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + 2t\mathbf{j},\ 0 \le t \le 1.$$

$\mathbf{C_2}$: is the line segment joining (0, 2, 0) and (0, 0, 2). Thus, it can parametrized as

$$\mathbf{r}(t) = (2 - 2t)\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 1.$$

$\mathbf{C_3}$: is the line segment joining (0, 0, 2) and (1, 0, 0). Thus, it can parametrized as

$$\mathbf{r}(t) = t\mathbf{i} + (2 - 2t)\mathbf{k},\ 0 \le t \le 1.$$

Then

$$
\begin{aligned}
L.H.S = \oint_C \mathbf{F}.\mathrm{d}\mathbf{r} &= \int_0^1 [(1-t)\mathbf{i} + 2t\mathbf{j}].(-\mathbf{i}+2\mathbf{j})\,\mathrm{d}t + \int_0^1 [(2-2t)\mathbf{j}].(-2\mathbf{j}-2\mathbf{k})\,\mathrm{d}t + \int_0^1 (t\mathbf{i}).(\mathbf{i}-2\mathbf{k})\,\mathrm{d}t \\
&= \int_0^1 (5t-1)\,\mathrm{d}t + \int_0^1 (4t-4)\,\mathrm{d}t + \int_0^1 t\,\mathrm{d}t \\
&= \frac{3}{2} - 2 + \frac{1}{2} = 0.
\end{aligned}
$$

R.H.S: Note that curl $\mathbf{F} = xz\mathbf{i} - yz\mathbf{j}$ and $z = 2 - 2x - y = g(x,y)$. Then

$$
\begin{aligned}
R.H.S = \iint_S \mathrm{curl}\mathbf{F}.\mathrm{d}\mathbf{S} &= \iint_{x,y\geq 0 \text{ and } 2-2x-y\geq 0} -x(2-2x-y)g_x + y(2-2x-y)g_y\,\mathrm{d}\mathbf{A} \\
&= \int_0^1 \int_0^{2-2x} (4x - 4y^2 - 2y + y^2)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 (4x(2-2x) - \frac{4}{3}(2-2x)^3 - (2-2x)^2 + \frac{(2-2x)^3}{3})\,\mathrm{d}x \\
&= \left[\frac{4}{3}x^4 - 4x^3 + 4x^2 - \frac{4}{3}x\right]_0^1 = 0.
\end{aligned}
$$

The fact that R.H.S = L.H.S verifies the Stokes' theorem.

18. **Solution:**(13.9 P 2)

    NOT COVERED IN EXAM.

19. **Solution:**(13.9 P 3)

    NOT COVERED IN EXAM.

20. **Solution:**(13.9 P 4)

    NOT COVERED IN EXAM.

21. **Solution:**(13.9 P 10)

    NOT COVERED IN EXAM.

22. **Solution:**(13.9 P 14)

    NOT COVERED IN EXAM.