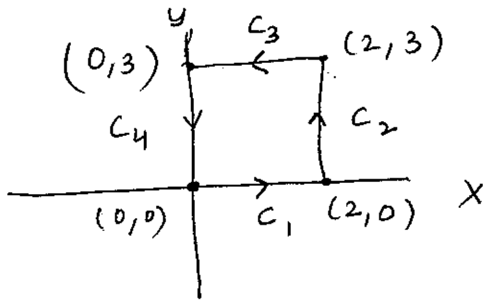
Math 21-259 Calculus in 3D
Homework 14 Solution
Spring 2011

1. **Solution:** We need to evaluate $\oint_C xy^2\,\mathrm{d}x + x^3\,\mathrm{d}y$ where C is the rectangle with vertices (0, 0), (2, 0), (2, 3), and (0, 3) directly and using Green's theorem. First of all, we sketch the curve C and divide it into smooth curves. Here, we have four smooth curves which we orient in the counter-clockwise direction and label by C_1, C_2, C_3, and C_4 as follows:



(a) **Direct Calculation:** Note, $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$. Thus, to compute the required integral we need to parametrize each of the curves C_i.

On C_1: $0 \le x \le 2$ and $y = 0$. This implies that $xy^2\,\mathrm{d}x + x^3\,\mathrm{d}y = x(0)^2\,\mathrm{d}x + x^3\,\mathrm{d}0 = 0$ and thus, $\int_{C_1} xy^2\,\mathrm{d}x + x^3\,\mathrm{d}y = 0$.

On C_2: $x = 2$ and $0 \le y \le 3$. This implies that $xy^2\,\mathrm{d}x + x^3\,\mathrm{d}y = 2(y)^2\,\mathrm{d}2 + 2^3\,\mathrm{d}y = 8\,\mathrm{d}y$ and thus, $\int_{C_2} xy^2\,\mathrm{d}x + x^3\,\mathrm{d}y = \int_0^3 8\,\mathrm{d}y = 24$.

On C_3: x is from 2 to 0 (I cannot write it using inequalities) and $y = 3$. This implies that $xy^2\,\mathrm{d}x + x^3\,\mathrm{d}y = x(3)^2\,\mathrm{d}x + x^3\,\mathrm{d}3 = 3x^2\,\mathrm{d}x$ and thus, $\int_{C_3} xy^2\mathrm{d}x + x^3\,\mathrm{d}y = \int_2^0 9x\,\mathrm{d}x = \left[9\frac{x^2}{2}\right]_2^0 = -18$.

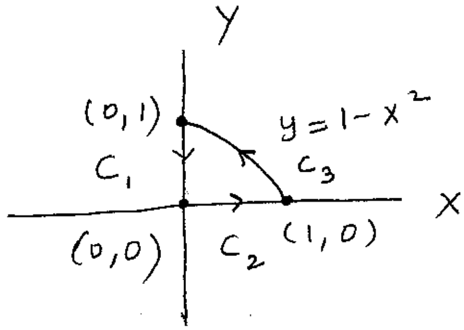
On C_4: $x = 0$ and y is from 3 to 0. This implies that $xy^2\,\mathrm{d}x + x^3\,\mathrm{d}y = 0^2 y\,\mathrm{d}0 + 0^3\,\mathrm{d}y = 0$ and thus, $\int_{C_4} \mathrm{d}x + x^3\,\mathrm{d}y = 0$.

Finally, $\oint_C xy^2\,\mathrm{d}x + x^3\,\mathrm{d}y = 0 + 24 - 18 + 0 = 6$.

(b) **Green's theorem:** Note that we can apply Green's theorem since we are given a closed curve and a function that is differentiable everywhere. According to the Green's theorem,

$$\begin{aligned}\oint_C xy^2\,\mathrm{d}x + x^3\,\mathrm{d}y &= \iint_{\text{Rectangle}} [\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(xy^2)]\,\mathrm{d}A \\ &= \int_0^2 \int_0^3 3x^2 - 2xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 (9x^2 - 9x)\,\mathrm{d}x = 24 - 18 = 6.\end{aligned}$$

2. **Solution:** We need to evaluate $\oint_C x\,\mathrm{d}x + y\,\mathrm{d}y$ where C is the join of the line segments from (0, 1) to (0, 0) and from (0, 0) to (1, 0) and the parabola $y = 1 - x^2$ from (1, 0) to (0, 1). First of all, we sketch the curve C and divide it into smooth curves. Here, we have three smooth curves which we orient in the counter-clockwise direction and label by C_1, C_2, and C_3 as follows:



(a) **Direct Calculation:** Note, $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$. Thus, to compute the required integral we need to parametrize each of the curves C_i.

On C_1: $x = 0$ and y is from 1 to 0. This implies that $x\,\mathrm{d}x + y\,\mathrm{d}y = y\,\mathrm{d}y$ and thus, $\int_{C_1} x\mathrm{d}x + y\,\mathrm{d}y = \int_1^0 y\,\mathrm{d}y = -\frac{1}{2}$.

On C_2: $0 \le x \le 1$ and $y = 0$. This implies that $x\,\mathrm{d}x + y\,\mathrm{d}y = x\,\mathrm{d}x$ and thus, $\int_{C_2} x\,\mathrm{d}x + y\,\mathrm{d}y = \int_0^1 x\,\mathrm{d}x = \frac{1}{2}$.

On C_3: x is from 1 to 0 and $y = 1 - x^2$. This implies that $x\,\mathrm{d}x + y\,\mathrm{d}y = x\,\mathrm{d}x + (1 - x^2)(-2x)\mathrm{d}x = (-x + 2x^3)\,\mathrm{d}x$ and thus, $\int_{C_3} x\,\mathrm{d}x + y\,\mathrm{d}y = \int_1^0 (-x + 2x^3)\,\mathrm{d}x = \left[-\frac{x^2}{2} + \frac{2}{4}x^4\right]_1^0 = \frac{1}{2} - \frac{1}{2} = 0.$

Finally, $\oint_C x\,\mathrm{d}x + y\,\mathrm{d}y = -\frac{1}{2} + \frac{1}{2} + 0 = 0.$

(b) **Green's theorem:** Note that we can apply Green's theorem since we are given a closed curve and a function that is differentiable everywhere. According to the Green's theorem,

$$\oint_C x\,\mathrm{d}x + y\,\mathrm{d}y = \iint_D [\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)]\,\mathrm{d}A = 0.$$

3. **Solution:** According to the Green's theorem for regions with holes, we see that

$$\begin{aligned}\int_{C_1+C_2} xe^{-2x}\,\mathrm{d}x + (x^4 + 2x^2y^2)\,\mathrm{d}y &= \iint_D \frac{\partial}{\partial x}(x^4 + 2x^2y^2) - \frac{\partial}{\partial y}(xe^{-2x})\,\mathrm{d}A \\ &= \iint_D (4x^3 + 4xy^2)\,\mathrm{d}A\end{aligned}$$

where C_1 is the boundary of the circle $x^2 + y^2 = 1$ and C_2 is the boundary of the circle $x^2 + y^2 = 4$, and D is the region bounded between C_1 and C_2.

To compute the above double integral, it is best to use polar coordinates. Thus,

$$\begin{aligned}
\iint_D (4x^3 + 4xy^2)\,\mathrm{d}A &= \iint_D 4x(x^2+y^2)\,\mathrm{d}A \\
&= \int_0^{2\pi}\int_1^2 4r\cos\theta(r^2)r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= [\sin\theta]_0^{2\pi}\left[\frac{4}{5}r^5\right]_1^2 = 0.
\end{aligned}$$

4. **Solution:** Note that the given curve is oriented clockwise. Thus, according to the Green's theorem

$$\begin{aligned}
\oint_C < y^2\cos x, x^2 + 2y\sin x > .\mathrm{dr} &= -\iint_D [\frac{\partial}{\partial x}(x^2+2y\sin x) - \frac{\partial}{\partial y}(y^2\cos x)]\,\mathrm{d}A \\
&= \iint_D -2x\,\mathrm{d}A = \int_0^2\int_0^{3x} -2x\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 [-2xy]_{y=0}^{y=3x}\,\mathrm{d}x = \int_0^2 -6x^2\,\mathrm{d}x = \left[-2x^3\right]_0^2 = -16.
\end{aligned}$$

5. **Solution:** We are given that $\mathbf{F} = \frac{x}{x^2+y^2+z^2}\,\mathbf{i} + \frac{y}{x^2+y^2+z^2}\,\mathbf{j} + \frac{z}{x^2+y^2+z^2}\,\mathbf{k}$.

(a)

$$\begin{aligned}
\mathrm{curl}(\mathbf{F}) = \nabla\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix} \\
&= \frac{1}{x^2+y^2+z^2}[(-2yz+2yz)\mathbf{i} - (-2xz+2xz)\mathbf{j} + (-2xy+2xy)\mathbf{k}] \\
&= \mathbf{0}.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{div}(\mathbf{F}) = \nabla.\mathbf{F} &= \frac{\partial}{\partial x}(\frac{x}{x^2+y^2+z^2}) + \frac{\partial}{\partial y}(\frac{y}{x^2+y^2+z^2}) + \frac{\partial}{\partial z}(\frac{z}{x^2+y^2+z^2}) \\
&= \frac{x^2+y^2+z^2-2x^2}{x^2+y^2+z^2} + \frac{x^2+y^2+z^2-2y^2}{x^2+y^2+z^2} + \frac{x^2+y^2+z^2-2z^2}{x^2+y^2+z^2} \\
&= \frac{1}{x^2+y^2+z^2}.
\end{aligned}$$

6. **Solution:** We are given that f is a scalar field and $\mathbf{F}$ is a vector field.

   (a) curl $f = \nabla\times f$ is meaningless just as a cross product of a scalar and a vector is undefined.

   (b) grad f is a vector field.

   (c) div $\mathbf{F}$ is scalar field.

   (d) curl(grad f) $= \nabla\times\nabla f$ is a vector field.

   (e) grad$\mathbf{F}$ is meaningless just as two vectors cannot be multiplied without having a dot or cross product between them.

(f) grad(div $\mathbf{F}$) is a vector field.

(g) div(grad $\mathbf{F}$) is a scalar field.

(h) grad(div f) is meaningless because f is a scalar field.

(i) curl(curl $\mathbf{F}$) is a vector field.

(j) div(div $\mathbf{F}$) is meaningless because $div\mathbf{F}$ is a scalar field.

(k) (grad f) $\times$ (div $\mathbf{F}$) is meaningless because div $\mathbf{F}$ is a scalar field.

(l) div(curl(grad f)) is a scalar field.

7. To determine if the given vector field $\mathbf{F} = e^x\,\mathbf{i} + \mathbf{j} + xe^z\,\mathbf{k}$ is conservative or not, we compute curl $\mathbf{F}$. Note that curl$\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^z & 1 & xe^z \end{vmatrix} = \mathbf{0}$. Thus, the given vector field $\mathbf{F} = \nabla f$ for some function of three variables f. To find f, we solve the following three equations.

$$\begin{aligned} f_x &= e^z && (1) \\ f_y &= 1 && (2) \\ f_z &= xe^z. && (3) \end{aligned}$$

Integrate (1) with respect to x, $f(x,y,z) = xe^z + g(y,z)$ and $f_y = g_y(y,z)$ but $f_y = 1$ from (2). From this it follows that $g_y = 1 \Rightarrow g(y,z) = y + h(z)$. Plugging this back in f yields $f(x,y,z) = xe^z + y + h(z)$ and $f_z = xe^z + h'(z) = xe^z$(from (3)). Thus, $h'(z) = 0 \Rightarrow h(z) = C$ and hence, $f(x,y,z) = xe^z + y + C$.