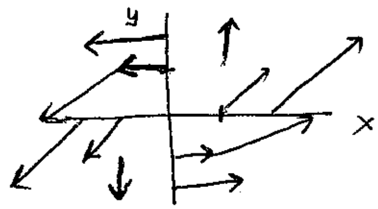
Math 21-259 Calculus in 3D
Homework 13 Solution
Spring 2011

1. **Solution:** We are given that $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + x\mathbf{j}$. Note that the vectors along the line $y = x$ are vertical with length $|x|$ and the vector along $x = 0$ are horizontal with length $|y|$.

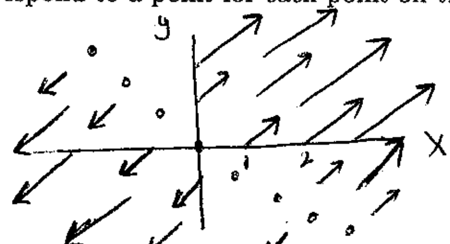


2. **Solution:** The vector function $\mathbf{F}(x, y) =< y, 1/x >$ corresponds to graph III. Note that the vectors in the field have the shape of a hyperbola. Moreover all the vectors in the first quadrant have positive $x-$ and $y-$components, in the second quadrant all vectors have positive $x-$components and negative $y-$components, in the third quadrant all vectors have negative $x-$ and $y-$components, and in the fourth quadrant all vectors have negative $x-$components and positive $y-$components. Note that the vectors become longer as the value of x gets smaller, that is, as approach the $y-$axis.

3. **Solution:** The gradient of the function $f(x, y) = \frac{1}{4}(x + y)^2$ is given by

$$\nabla \mathrm{f} =< \frac{1}{2}(x + y), \frac{1}{2}(x + y) > .$$

Note that the length of each is equal to $\frac{1}{\sqrt{2}}|x + y|$. Along the line $y = -x$, the vectors have zero length, i.e., they correspond to a point for each point on the line.



4. **Solution:** By plugging $x = t^4$, $y = t^3$ in the integral $\int_C \frac{y}{x}\,\mathrm{d}s$, we get

$$\begin{aligned}
\int_{1/2}^{1} \frac{t^3}{t^4} | < 4t^3, 3t^2 > | \mathrm{d}t &= \int_{1/2}^{1} \frac{1}{t}\sqrt{16t^6 + 9t^4}\,\mathrm{d}t \\
&= \int_{1/2}^{1} t\sqrt{16t^2 + 9}\,\mathrm{d}t \overset{u=16t^2+9}{=\!=\!=} \int_{4+9}^{16+9} \frac{1}{32}\sqrt{u}\,\mathrm{d}u \\
&= \left[\frac{2}{32(3)}u^{3/2}\right]_{u=13}^{u=25} = \frac{1}{48}(125 - 13\sqrt{13}).
\end{aligned}$$

5. **Solution:** Note that the given curve is a piecewise smooth curve. Thus, it can be split into two smooth curves, namely, an arc of the given circle (C_1) and a line segment (C_2) joining (-1, 0) and (-2, 3). Then $\int_C \sin x\,dx + \cos y\,dy = \int_{C_1} \sin x\,dx + \cos y\,dy + \int_{C_2} \sin x\,dx + \cos y\,dy$.

Parametrize C_1 by using $x = \cos t,\ y = \sin t,\ 0 \le t \le \pi$. Then $dx = -\sin t\,dt$ and $dy = \cos t\,dt$. Using this in the integral we get

$$\begin{aligned}
\int_{C_1} \sin x\,dx + \cos y\,dy &= \int_0^{\pi} \sin(\cos t)(-\sin t\,dt) + \cos(\sin t)(\cos t\,dt) \\
&= \int_0^{\pi} \sin(\cos t)(-\sin t)\,dt + \int_0^{\pi} \cos(\sin t)(\cos t)\,dt \\
&\overset{u=\cos t,\ v=\sin t}{=\!=\!=\!=\!=} \int_{\cos 0}^{\cos \pi} \sin(u)\,du + \int_{\sin 0}^{\sin \pi} \cos(v)\,dv \\
&= [\cos u]_1^{-1} + int_0^0 \cos(v)\,dv = \cos(-1) - \cos(1) + 0 = 0.
\end{aligned}$$

Parametrize C_2 by using $x = -1 - t,\ y = 3t,\ 0 \le t \le 1$. Then $dx = -dt$ and $dy = 3\,dt$. Using this in the integral we get

$$\begin{aligned}
\int_{C_2} \sin x\,dx + \cos y\,dy &= \int_0^1 \sin(-1-t)(-dt) + \cos(3t)(3\,dt) \\
&= [-\cos(-1-t) + \sin(3t)]\,dt]_0^1 \\
&= [-\cos(-2) + \sin(3) + \cos(-1) - \sin(0)] \\
&\overset{\cos(-\theta)=\cos\theta}{=\!=\!=\!=} \cos 1 - \cos 2 + \sin 3.
\end{aligned}$$

Thus, the value of the required integral $= \cos 1 - \cos 2 + \sin 3$.

6. Let C be the given curve which can be parametrized by using $x = t$ and $y = t^2,\ -1 \le t \le 2$. Note that the work done by the force field $\mathbf{F}(x, y) = < x\sin y, y >$ over the curve C is given by

$$\begin{aligned}
W = \int_C < x\sin y, y > .\mathrm{dr} &= \int_{-1}^{2} < t\sin t^2, t^2 > . < 1, 2t > dt \\
&= \int_{-1}^{2} t\sin t^2 + 2t^3\,dt \\
&= \int_{-1}^{2} t\sin t^2\,dt + \left[\frac{2t^4}{4}\right]_{-1}^{2} \\
&\overset{u=x^2}{=} \int_{(-1)^2}^{2^2} \frac{1}{2}\sin u\,du + [8 - \frac{1}{2}] \\
&= [\frac{1}{2}(\cos 1 - \cos 4)] + \frac{15}{2}.
\end{aligned}$$

7. For the given function, $P = xy\cos xy + \sin xy$, $Q = x^2\cos xy$. Note that $Q_x = -x^2y\sin xy + 2x\cos xy = P_y$ and $\mathbf{F}$ is defined everywhere. Hence, $\mathbf{F}$ is conservative, that is, there exists a function f such that $\mathbf{F} = \nabla f$. To find f, we need to solve the following two equations simultaneously.

$$f_x = xy\cos xy + \sin xy \quad \text{and} \qquad (1)$$

$$f_y = x^2\cos xy \qquad (2)$$

Note that it is easier to integrate (2) with respect to y instead of integrating (1) with respect to x. Therefore, by integrating (2) with respect to y, we get $f(x,y) = x\sin xy + g(x)$. We now take the partial derivative of this function f with respect to x and match it up the equation (1). Thus, we get $f_x = xy\cos xy + \sin xy + g'(x) = xy\cos xy + \sin xy$ which further implies that $g'(x) = 0$. Thus, we get $g(x) = C$ and $f(x,y) = x\sin xy + C$.

8. **Solution:** We are given that $\mathbf{F}$ is conservative where $P = \frac{y^2}{1+x^2}\mathbf{i}$ and $Q = 2y\arctan x\mathbf{j}$. To find f, we need to solve the equations $f_x = P = \frac{y^2}{1+x^2}$ and $f_y = Q = 2y\arctan x$. By integrating the first equation with respect to x, we get $f(x,y) = y^2\arctan x + g(y)$. Then $f_y = 2y\arctan x + g'(y) = 2y\arctan x$ (given) $\Rightarrow g'(y) = 0 \Rightarrow g(y) = C$. Thus, $f(x,y,z) = y^2\arctan x + C$.

9. **Solution:** To find the work done by the force field $\mathbf{F}(x,y) = e^{-y}\,\mathbf{i} - xe^{-y}\,\mathbf{j}$ in moving from P(0,1) to Q(2, 0), we need to calculate the line integral $\int_{\mathbf{PQ}} \mathbf{F}.\text{dr}$. First of all, we will quickly check if the given force field is conservative or not. Note that $P_y = -e^{-y} = Q_x$ which implies that $\mathbf{F} = \nabla f$ for some f. To find f, we solve $f_x = e^{-y}$ and $f_y = -xe^{-y}$. Note from the first equation, we get that $f(x,y) = xe^{-y} + g(y)$ and $f_y = -xe^{-y} + g'(y)$. But we are given that $f_y = -xe^{-y}$. Thus, by equating we get $g(y) = 0 \Rightarrow g(y) = C$. This yields $f(x,y) = xe^{-y} + C$. Finally, $\int_{\mathbf{PQ}} \mathbf{F}.\text{dr} = f(Q) - f(P) = (2e^{-0} + C) - (0e^{-1} + C) = 2$. Note that the constant C never counts in the calculation of line integral using Fundamental theorem of line integral just as in single variable calculus.

10. **Solution:** From Exercise 25, we see that if $\mathbf{F}$ is conservative (or path independent) then $P_y = Q_x$, $P_z = R_x$, $Q_z = R_y$. Thus, the contrapositive of the above statement states that if any of the conditions $P_y = Q_x$, $P_z = R_x$, $Q_z = R_y$ fail then the vector field $\mathbf{F}$ cannot be conservative, i.e., path independent. For our given problem, we have $P = y$, $Q = x$, $R = xyz$. Note that $P_y = 1 = Q_x$ but $P_z = 0 \neq R_x$. This implies that the given vector field $\mathbf{F}(x,y,z) =< y, x, xyz >$ cannot be conservative or path independent.