CARNIVAL HOMEWORK

1. This homework is given to introduce you to the concept of improper integrals for double integrals and also to illustrate that the double integrals can be effective in computing important single variable integrals.

"Look at the result of Q3."

2. You should compare this concept with the known concept of improper "single variable" integral.

3. This homewok contain 6 problems in total.

3. Each question is worth 2 points.

1. Section 12.3 Problem 30(a)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\,dA$$

<u>Note:</u>

$$\iint_{D_a} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^{a} \underline{e^{-r^2}\,r\,dr}\,d\theta$$

$u = r^2$

$du = 2r\,dr$

$$= \int_0^{2\pi}\int_0^{a^2} e^{-u}\,\frac{du}{2}\,d\theta$$

$$= \cancel{2}\pi\left[\frac{-e^{-u}}{\cancel{2}}\right]_0^{a^2} = \pi\left(1-e^{-a^2}\right)$$

$$I = \lim_{a\to\infty}\pi\left(1-\cancelto{0}{e^{-a^2}}\right) = \boxed{\pi}$$

limit

2. Section 12.3 Problem 30(b) Hint: Use Q1

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA$$

$$= \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\,dA$$

$$\boxed{e^{-(x^2+y^2)} = e^{-x^2}\cdot e^{-y^2}}$$

split up!

$$= \lim_{a\to\infty} \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \lim_{a\to\infty} \left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$$

But $\iint_{R^2} e^{-(x^2+y^2)}\, dA = \pi$ (from Q1)

Therefore, $\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \pi}$

3. Section 12.3 Problem 30(c) Hint: Use Q2

We proved in Q2 that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \pi$$

But $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-y^2}\,dy$ (simply the change of variable)

Therefore, $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

Hence proved

4. Section 12.3 Problem 30(d)   Hint: Use Q3

To show: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$

Consider, $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\,dx \underset{du = \frac{dx}{\sqrt{2}}}{\overset{u = \frac{x}{\sqrt{2}}}{=\!=}} \int_{-\infty}^{\infty} e^{-u^2}\sqrt{2}\,du$

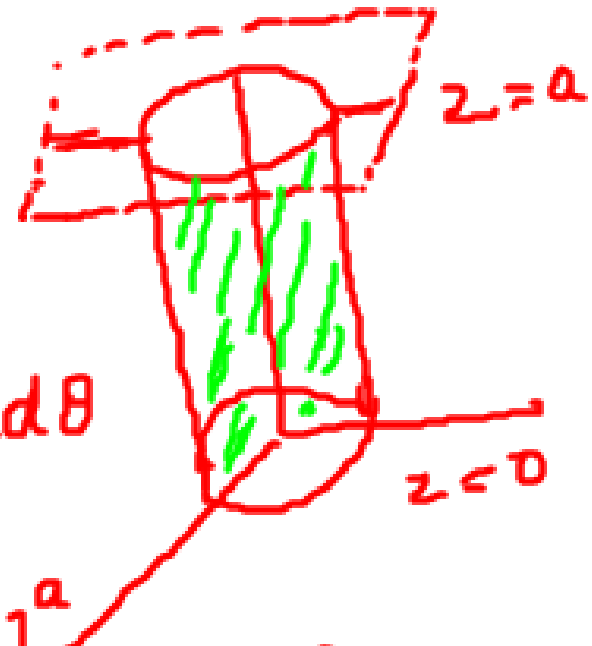
$$= \sqrt{2}\int_{-\infty}^{\infty} e^{-u^2}\,du$$

$$= \sqrt{2}\left(\sqrt{\pi}\right) \quad \text{(From Q3)}$$

$$= \boxed{\sqrt{2\pi}}$$

5. Evaluate $\displaystyle\iiint_{E_a} \frac{e^{-z}}{(x^2+y^2+1)^2}\, dV$ where E_a is the solid determined by the cylinder $x^2+y^2=a^2$ and the plane $z=0$ and $z=a$.

Sol^n We shall use cylindrical coordinates.



$$\iiint_{E_a}\frac{e^{-z}}{(1+x^2+y^2)^2}\,dV = \int_0^{2\pi}\int_0^a\int_0^a \frac{e^{-z}}{(1+r^2)^2}\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^a \frac{r}{(1+r^2)^2}\left[-e^{-z}\right]_0^a dr\,d\theta$$

↳ integral w.r.to z.

$$= \int_0^{2\pi}\int_0^{a} \frac{(1-e^{-a})\, r}{(1+r^2)^2}\, dr\, d\theta$$

$u = 1 + r^2$

$du = 2r\, dr$

$$= \int_0^{2\pi}\int_1^{1+a^2} \frac{(1-e^{-a})}{u^2}\, \frac{du}{2}\, d\theta$$

$$= \int_0^{2\pi} \frac{(1-e^{-a})}{2} \left[\frac{u^{-1}}{-1}\right]_1^{1+a^2} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1-e^{-a}}{2}\right)\left(1 - \frac{1}{1+a^2}\right) d\theta$$

$$= \pi\left(1-e^{-a}\right)\left(1 - \frac{1}{1+a^2}\right)$$

6. Find $\lim\limits_{a\to\infty} \iiint\limits_{E_a} \dfrac{e^{-z}}{(1+x^2+y^2)^2}\, dV$. **Hint: Use Q5.**

Sol^n From Q5, $\iiint\limits_{E_a} \dfrac{e^{-z}}{(1+x^2+y^2)^2}\, dV = \pi(1-e^{-a})\left(1-\dfrac{1}{1+a^2}\right)$

$$\lim_{a\to\infty} \iiint\limits_{E_a} \frac{e^{-z}}{(1+x^2+y^2)^2}\, dV = \lim_{a\to\infty} \pi \left(1-\cancel{e^{-a}}^{\,0}\right)\left(1-\cancel{\frac{1}{1+a^2}}^{\,0}\right)$$

$$= \boxed{\pi} \text{ Ans}$$