Math 21-259 Calculus in 3D
Homework 11 Solution
Spring 2011

1. **Solution:**

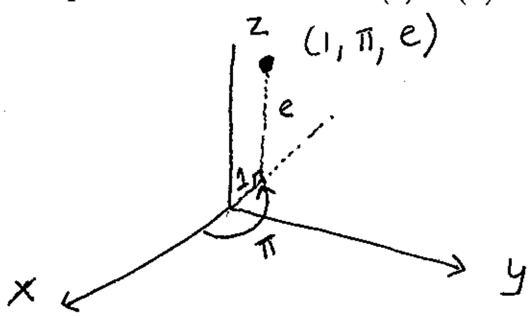
Rectangular Coordinates	Cylindrical Coordinates
(x, y, z)	(r, θ, z)

where $x = r\cos\theta,\ y = r\sin\theta,\ z = z$.

(a) Convert Cylindrical coordinates into Rectangular coordinates.

Rectangular Coordinates	Cylindrical Coordinates
$(-1, 0, e)$	$(1, \pi, e)$

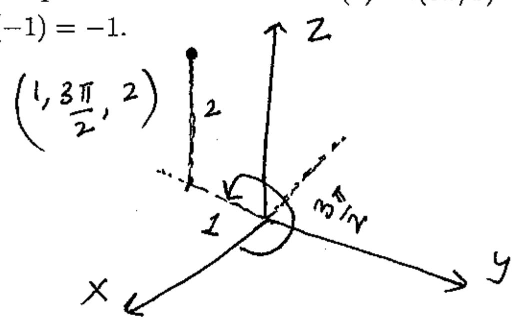
because $r = 1,\ \theta = \pi$ which implies that $x = r\cos\theta = (1)\cos(\pi) = (1)(-1) = -1$ and $y = r\sin\theta = (1)\sin(\pi) = 0$.



(b) Convert Cylindrical coordinates into Rectangular coordinates.

Rectangular Coordinates	Cylindrical Coordinates
$(0, -1, 2)$	$(1, 3\pi/2, 2)$

because $r = 1,\ \theta = 3\pi/2$ which implies that $x = r\cos\theta = (1)\cos(3\pi/2) = (1)(0) = 0$ and $y = r\sin\theta = (1)\sin(3\pi/2) = (1)(-1) = -1$.

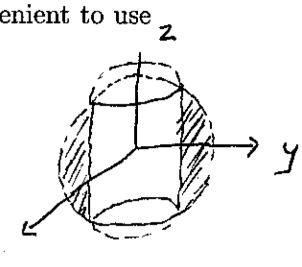


2. **Solution:** Note that in both Cylindrical and Spherical Coordinate system, $\theta = \pi/3$ represents a half-plane including the z-axis and intersecting the xy-plane in the half-line $y = \sqrt{3}x,\ x > 0$.

3. **Solution:** Note that $z = r$ represents a cone $z = \sqrt{x^2 + y^2}$ that opens upwards in the space. Thus $r \leq z \leq 2$ indicates that we are looking at the region that is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 2$. Additional condition $0 \leq \theta \leq \frac{\pi}{2}$ simply restricts the region to the first octant.

4. **Solution:** Given the fact that the problem involves the cylinder, it seems convenient to use cylindrical coordinates.
Step I. Sketch the given solid to get an idea.

Required region = shaded region



Step II. Convert the given equations in cylindrical coordinate system.

Cylinder	Rectangular Coordinates	Cylindrical Coordinates
	$x^2 + y^2 = 1$	$r = 1$

Sphere	Rectangular Coordinates	Cylindrical Coordinates
	$x^2 + y^2 + z^2 = 4$	$r^2 + z^2 = 4$

Step III. Describe the Region.

Note that the given solid lies within the cylinder $r = 1$ and is bounded above by the upper part of the sphere $z = \sqrt{4 - r^2}$ and below by the lower part of the sphere $z = -\sqrt{4 - r^2}$.

Step IV. Set up the integral and Compute it.

The volume of the above described solid region is given by

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz dr d\theta \\
&= \int_0^{2\pi} \int_0^1 2r\sqrt{4 - r^2} dr d\theta \\
&= \int_0^{2\pi} \int_4^3 -\sqrt{u} du d\theta \qquad [\text{Use } u = 4 - r^2 and du = -2r dr] \\
&= \int_0^{2\pi} -\frac{2}{3} u^{3/2}|_{u=4}^{u=3} d\theta \\
&= \int_0^{2\pi} [16/3 - 2\sqrt{3}] d\theta = 2\pi[16/3 - 2\sqrt{3}].
\end{aligned}
$$

5. We are given that $-3 \le x \le 3$, $0 \le y \le \sqrt{9-x^2}$, and $0 \le z \le 9 - x^2 - y^2$. The limits for z indicate that the region of integration is the region above xy-plane $z = 0$ and below the paraboloid $z = 9 - x^2 - y^2$. The limits for x and y together describes the projection of the solid region in the xy-plane which describes the upper half of a circle of radius 3 in the xy-plane centered at (0, 0). Thus,

$$\begin{aligned}
\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\,\mathrm{d}z\mathrm{d}y\mathrm{d}x &= \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2} \sqrt{r^2}r\,\mathrm{d}z\mathrm{d}r\mathrm{d}\theta \\
&= \int_{0}^{\pi}\int_{0}^{3}(9-r^2)r^2\,\mathrm{d}r\mathrm{d}\theta \\
&= \int_{0}^{\pi}\left[3r^3 - \frac{r^5}{5}\right]_{r=0}^{r=3}\mathrm{d}\theta \\
&= \int_{0}^{\pi}(3(3^3) - \frac{3^5}{5})\,\mathrm{d}\theta = \frac{162}{5}\pi.
\end{aligned}$$

6. **Solution:**

Rectangular Coordinates	Spherical Coordinates
(x, y, z)	(ρ, θ, ϕ)

where $x = (\rho\sin\phi)\cos\theta,\ y = (\rho\sin\phi)\sin\theta,\ z = \rho\cos\phi.$

(a) Convert Rectangular coordinates into Spherical coordinates.

Rectangular Coordinates	Spherical Coordinates
$(0, \sqrt{3}, 1)$	$(2, \frac{\pi}{3}, \frac{\pi}{2})$

because $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0+3+1} = 2$, $\cos\phi = \frac{z}{\rho} = \frac{1}{2}$ which implies that $\phi = \frac{\pi}{3}$, and $\cos(\theta) = \frac{x}{\rho\sin\phi} = \frac{0}{2\sin(\pi/3)} = 0 \Rightarrow \theta = \frac{\pi}{2}$, since $y > 0$.

(b) Convert Rectangular coordinates into Spherical coordinates.

Rectangular Coordinates	Cylindrical Coordinates
$(-1, 1, \sqrt{6})$	$(2\sqrt{2}, 3\pi/4, \pi/6)$

because $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+1+6} = 2\sqrt{2}$, $\cos\phi = \frac{z}{\rho} = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$ which implies that $\phi = \frac{\pi}{6}$, and $\cos(\theta) = \frac{x}{\rho\sin\phi} = \frac{-1}{2\sqrt{2}\sin(\pi/6)} = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$, since $y > 0$.

7. **Solution:** Note that $2 \le \rho \le 3$ represents the solid region between and including the spheres of radii 2 and 3, centered at the origin and the fact that $\frac{\pi}{2} \le \phi \le \pi$ restricts the solid to that portion on or below the xy−plane.

8. **Solution:** Note that the given solid **E** in spherical coordinates can be described as follows:

$$E = \{(\rho, \theta, \phi) | 0 \le \rho \le 3,\ 0 \le \theta \le \pi/2,\ \ 0 \le \phi \le \pi/2\}.$$

Thus,

$$\begin{aligned}
\iiint_E e^{\sqrt{x^2+y^2+z^2}}\, \mathrm{d}V &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 e^{\sqrt{\rho^2}} \rho^2 \sin\phi \,\mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta \\
&= \left(\int_0^{\pi/2} \mathrm{d}\theta\right)\left(\int_0^{\pi/2} \sin\phi\, \mathrm{d}\phi\right)\left(\int_0^3 e^{\rho}\rho^2\, \mathrm{d}\rho\right)_{\text{[Integrate by parts twice]}} \\
&= (\pi/2)\, [-\cos\phi]_0^{\pi/2}\, [(\rho^2 - 2\rho + 2)e^{\rho}]_0^3 = \frac{\pi}{2}(5e^3 - 2).
\end{aligned}$$

9. **Solution:** We shall use spherical coordinates to compute the volume of the given solid. In spherical coordinates, sphere $x^2+y^2+z^2 = 4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2+y^2}$ is equivalent to $\phi = \pi/4$. Thus, we can set up the volume of the given solid as

$$\begin{aligned}
V &= \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\phi \\
&= \left(\int_{\pi/4}^{\pi/2} \sin\phi\, \mathrm{d}\phi\right)\left(\int_0^{2\pi} \mathrm{d}\theta\right)\left(\int_0^2 \rho^2\, \mathrm{d}\rho\right) \\
&= [\cos\phi]_{\pi/4}^{\pi/2}\, [\theta]_0^{2\pi} \left[\frac{\rho^3}{3}\right]_0^2 = \frac{8\sqrt{2}}{3}\pi.
\end{aligned}$$

10. **Solution:** We are given that $x = e^{u-v},\ y = e^{u+v},\ z = e^{u+v+w}$. Recall, Jacobian is given by

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \begin{pmatrix} e^{u-v} & -e^{u-v} & 0 \\ e^{u+v} & e^{u+v} & 0 \\ e^{u+v+w} & e^{u+v+w} & e^{u+v+w} \end{pmatrix} \\
&= e^{u+v+w} \begin{pmatrix} e^{u-v} & -e^{u-v} \\ e^{u+v} & e^{u+v} \end{pmatrix} = e^{u+v+w}[e^{u-v}e^{u+v} + e^{u-v}e^{u+v}] \\
&= e^{u+v+w}[2e^{2u}] = 2e^{3u+v+w}.
\end{aligned}$$

11. **Solution:** Note that the given parallelogram is bounded by the lines $x - y = -4,\ x - y = 4,\ 3x + y = 0,\ 3x + y = 8$. This suggests that the change of variable $u = x - y$ and $v = 3x + y$ will significantly simplify the given integral. Note that $u = x - y$ and $v = 3x + y$ can be solved simultaneously to get the given change of variable $x = \frac{1}{4}(u + v),\ y = \frac{1}{4}(v - 3u)$. To set

up the integral, we need Jacobian, $\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{pmatrix} = \frac{1}{4}$. Thus, integral is given by

$$\begin{aligned}
\iint_R (4x+8y)\,\mathrm{d}A &= \iint_{\text{New Rectangle}} [4.\frac{1}{4}(u+v) + 3.\frac{1}{4}(v-3u)] \left|\frac{1}{4}\right| \mathrm{d}A' \\
&\quad [\text{New Rectangle} : -4 \le u \le 4, 0 \le v \le 8] \\
&= \int_{-4}^{4}\int_{0}^{8} (3v-5u) \left|\frac{1}{4}\right| \mathrm{d}v\,\mathrm{d}u = \frac{1}{4}\int_{-4}^{4} \left[\frac{3}{2}v^2 - 5uv\right]_{v=0}^{v=8} \mathrm{d}u \\
&= \frac{1}{4}\int_{-4}^{4} (96-40u)\,\mathrm{d}u = \frac{1}{4}[96u - 20u^2]_{-4}^{4} = 192.
\end{aligned}$$

12. **Solution:** By letting $u = x+y,\ v = x-y$, and solving these equations simultaneously we get that $u = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$. Then $\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -\frac{1}{2}$ and the new rectangle in the $uv-$plane is enclosed by the lines $u = 0,\ u = 3,\ v = 0$, and $v = 2$. Thus the integral is given by

$$\begin{aligned}
\iint_R (x+y)e^{x^2-y^2}\,\mathrm{d}A &= \iint_{\text{New Rectangle}} ue^{uv} \left|-\frac{1}{2}\right| \mathrm{d}A' \\
&= \int_0^3\int_0^2 (ue^{uv}) \left|-\frac{1}{2}\right| \mathrm{d}v\,\mathrm{d}u = \frac{1}{2}\int_0^3 [e^{uv}]_{v=0}^{v=2}\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^3 (e^{2u}-1)\,\mathrm{d}u = \frac{1}{2}[\frac{1}{2}e^{2u} - u]_0^3 = \frac{1}{2}(\frac{1}{2}e^6 - 3 - \frac{1}{2}) = \frac{1}{4}(e^6 - 7).
\end{aligned}$$