Math 21-259 Calculus in 3D
Homework 10 Solution
Spring 2011

1. **Solution:** Note that the given region $\mathbf{R}$ is the part of the annular ring(with inner radius 1 and outer radius 2) that lies in the second and the third quadrant. Thus,

$$\begin{aligned}\iint_{\mathbf{R}} (x+y)\,\mathrm{d}A &= \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta)r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_{\pi/2}^{3\pi/2} (\cos\theta+\sin\theta)\,\mathrm{d}\theta \int_1^2 r\,\mathrm{d}r \\ &= [-\sin\theta+\cos\theta]_{\pi/2}^{3\pi/2}\left[\frac{r^2}{2}\right]_1^2 = 2(2-1/2) = 3.\end{aligned}$$

2. **Solution:** By changing the given integral into polar coordinates, we get

$$\begin{aligned}\iint_{\mathbf{R}} ye^x\,\mathrm{d}A &= \int_0^{\pi/2}\int_0^5 r\sin\theta e^{r\cos\theta} r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^5\int_0^{\pi/2} r^2\sin\theta e^{r\cos\theta}\,\mathrm{d}\theta\mathrm{d}r \qquad \text{[Switch the order of integration]} \\ &= \int_0^5\int_{\cos 0}^{\cos\pi/2} r^2 e^{ru}\,(-\mathrm{d}u)\mathrm{d}r \qquad \text{[Use substitution } u=\cos\theta\text{]} \\ &= \int_0^5 \left[r^2\frac{e^{ru}}{r}\right]_{u=1}^{u=0} \mathrm{d}r \\ &= \int_0^5 r^2\frac{1}{r} - \frac{e^r}{r}\,\mathrm{d}r \\ &= \int_0^5 r(1-e^r)\,\mathrm{d}r \\ &= \left[re^r - e^r - \frac{1}{2}r^2\right]_0^5 \qquad \text{[Use Integration by parts]} \\ &= 4e^5 - \frac{23}{2}.\end{aligned}$$

3. **Solution:** Note that the height of the given solid is given by $2\sqrt{16-x^2-y^2}$ and the projection of the given region in the xy-plane is the annular ring with inner radius 2 and outer radius 4. Thus,

$$\begin{aligned}
\iint_{4\le x^2+y^2\le 16} 2\sqrt{16-x^2-y^2}\,\mathrm{d}A &= \int_0^{2\pi}\int_2^4 2\sqrt{16-r^2}\; r\,\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_{16-2^2}^{16-4^2} -\sqrt{u}\,\mathrm{d}u\mathrm{d}\theta \qquad \text{[Use substitution } u = 16-r^2\text{]} \\
&= \int_0^{2\pi}\left[-\frac{u^{3/2}}{3/2}\right]_{u=12}^{u=0} \mathrm{d}\theta \\
&= \int_0^{2\pi}\left[\frac{2}{3}(12)^{3/2}\right] \mathrm{d}\theta \\
&= \frac{2}{3}(2\pi)(12)^{3/2} = 32\sqrt{3}\pi.
\end{aligned}$$

4. **Solution:** Note that the solid is bounded below by the paraboloid and above by the plane $z = 7$, thus the height of the solid is given by $7-(1+2x^2+2y^2)$. The shadow of the given region in the xy-plane can be found by solving the following inequality:

$$z_{\text{bottom}} \le z_{\text{top}} \iff 1-2x^2-2y^2 \le 7 \iff x^2+y^2 = 3.$$

Also, we are restricted to the first quadrant only. Thus the required integral is given by

$$\begin{aligned}
\iint_{x^2+y^2\le 3,\ x\ge 0,\ y\ge 0} [7-(1+2x^2+2y^2)]\,\mathrm{d}A &= \int_0^{\pi/2}\int_0^{\sqrt{3}} [7-(1+2r^2)]r\,\mathrm{d}r\mathrm{d}\theta \\
&= \left(\int_0^{\pi/2} \mathrm{d}\theta\right)\left(\int_0^{\sqrt{3}} (6-2r^2)r\,\mathrm{d}r\right) \\
&= (\pi/2)\left[3r^2-\frac{2}{4}r^4\right]_0^{\sqrt{3}} = (\pi/2)(9-9/2) = \frac{9}{4}\pi.
\end{aligned}$$

5. **Solution:** The first step is to sketch the region that is determined by the bounds in the given integral. Note that $0 \le x \le 2$ and $0 \le y \le \sqrt{2x-x^2}$ describes the semicircular region with radius 1 and center (1, 0) that lies above the x-axis. The same region can be described in

polar coordinates by the inequalities: $0 \le r \le 2\cos\theta,\ 0 \le \theta \le \pi/2$. Thus,

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,\mathrm{dydx} &= \int_0^{\pi/2}\int_0^{2\cos\theta} [r]\ r\,\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\ cos\theta} \mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{(2\cos\theta)^3}{3}\,\mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{8}{3}(\cos\theta)^2\cos\theta\,\mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{8}{3}(1-\sin^2\theta)\cos\theta\,\mathrm{d}\theta \\
&= \int_{\sin 0}^{\sin\pi/2} \frac{8}{3}(1-u^2)\,\mathrm{d}u \qquad [\text{Use u} - \text{sub},\ \text{u} = \sin\theta] \\
&= \left[\frac{8}{3}(u - \frac{u^3}{3})\right]_0^1 = \frac{8}{3}(1-1/3) = \frac{16}{9}.
\end{aligned}$$

6. **Solution:** Let $I = \int_0^1 \int_x^{2x} \int_0^y 2xyz\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$.

$$\begin{aligned}
I &= \int_0^1 \int_x^{2x} \left[xyz^2\right]_{z=0}^{z=y} \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_x^{2x} xy^3\,\mathrm{dydx} \\
&= \int_0^1 \left[\frac{y^4}{4}x\right]_{y=x}^{y=2x} \mathrm{dx} \\
&= \int_0^1 \frac{15}{4}x^5\,\mathrm{dx} \\
&= \left[\frac{5}{8}x^6\right]_0^1 = \frac{5}{8}.
\end{aligned}$$

7. **Solution:** Let $I = \iiint_E yz\cos(x^5)\,\mathrm{dV}$, where $E : 0 \le x \le 1, 0 \le y \le x, x \le z \le 2x$. Note

$$\begin{aligned}
I &= \int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_0^x \frac{3}{2}x^2 y\cos(x^5)\,\mathrm{dydx} \\
&= \int_0^1 \int_0^x \left[\frac{3}{4}x^2y^2\cos(x^5)\right]_{y=0}^{y=x} \mathrm{dx} \\
&= \int_0^1 \frac{3}{4}x^4\cos(x^5)\,\mathrm{dx} \\
&= \int_{0^5}^{1^5} \frac{3}{4(5)}\cos u\,\mathrm{d}u \qquad [\text{Use u} - \text{sub},\ \text{u} = \text{x}^5] \\
&= \left[\frac{3}{20}\sin u\right]_0^1 = \frac{3}{20}\sin 1.
\end{aligned}$$

8. **Solution:** Let $I = \iiint_E xz\,\mathrm{d}V$, where E is the solid tetrahedron with vertices (0, 0, 0), (0, 1, 0), (1, 1, 0), and (0, 1, 1). Note that this solid can easily be treated as type 1, type 2 or type 3 region. Suppose we decide to write the integral in the order $\mathrm{d}x\mathrm{d}z\mathrm{d}y$. To find the bounds for y and z, we shall need the projection of the tetrahedron in the yz-plane. Note that the projection of the tetrahedron is the yz-plane is the triangle with vertices (0, 0), (1, 1), (1, 0). Thus, we can describe the given solid as $E = \{(x, y, z) | 0 \leq x \leq y - z,\ 0 \leq y \leq 1,\ 0 \leq z \leq y, \}$ which allows us to write the integral as follows:

$$\begin{aligned} I &= \int_0^1 \int_0^y \int_0^{y-z} xz\,\mathrm{dxdzdy} \\ &= \int_0^1 \int_0^y \left[\frac{x^2}{2} z\right]_{x=0}^{x=y-z} \mathrm{dzdy} \\ &= \int_0^1 \int_0^y \frac{(y-z)^2}{2} z\,\mathrm{dzdy} \\ &= \int_0^1 \int_0^y \frac{1}{2}(zy^2 + z^3 - 2yz^2)\,\mathrm{dzdy} \\ &= \int_0^1 \left[\frac{y^2 z^2}{4} + \frac{z^4}{8} - \frac{yz^3}{3}\right]_{z=0}^{z=y} \mathrm{dy} \\ &= \int_0^1 \left[\frac{y^4}{4} + \frac{y^4}{8} - \frac{y^4}{3}\right] \mathrm{dy} \\ &= \left[(\frac{1}{4} + \frac{1}{8} - \frac{1}{3})\frac{y^5}{5}\right]_{y=0}^{y=1} = \frac{1}{24(5)} = \frac{1}{120}. \end{aligned}$$

9. **Solution:** Note that the given solid is bounded below by the xy-plane($z = 0$) and above by the plane $z = 4$ and its projection in the xy-plane is the region bounded by the parabola $y = x^2$ and the line $y = 9$. Thus the volume of the given solid is given by

$$\begin{aligned} V &= \int_{-3}^3 \int_{x^2}^9 \int_0^4 \mathrm{dzdydx} \\ &= \int_{-3}^3 \int_{x^2}^9 4\,\mathrm{dydx} \\ &= \int_{-3}^3 4(9 - x^2)\,\mathrm{dx} \\ &= 4\left[9x - \frac{x^3}{3}\right]_{x=-3}^{x=3} \\ &= 4[(9(3) - 3^2) - (9(-3) - (-3^2))] = 144. \end{aligned}$$

10. **Solution:** Note that the given solid is the tetrahedron with vertices (0, 0, 0), (0, 2, 0), (0, 2, 2), and (1, 2, 0). Sketch the projections of this solid on all three $xy-$, $yz-$, and $xz-$planes. If D_1, D_2, and D_3 are the projections of the given solid on the $xy-$, $yz-$, and $xz-$ planes

respectively, then

$$\begin{aligned}
D_1 &= \{(x,y)|0\le x\le 1,\ 2x\le y\le 2\} = \{(x,y)|0\le y\le 2,\ 0\le x\le y/2\}\\
D_2 &= \{(y,z)|0\le y\le 2,\ 0\le z\le y\} = \{(y,z)|0\le z\le 2,\ z\le y\le 2\}\\
D_3 &= \{(x,z)|0\le x\le 1,\ 0\le z\le 2-2x\} = \{(x,z)|0\le z\le 2,\ 0\le x\le (2-z)/2\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E &= \{(x,y,z)|0\le x\le 1,\ 2x\le y\le 2,\ 0\le z\le y-2x\}\\
E &= \{(x,y,z)|0\le y\le 2,\ 0\le x\le y/2,\ 0\le z\le y-2x\}\\
E &= \{(x,y,z)|0\le y\le 2,\ 0\le z\le y,\ 0\le x\le (y-z)/2\}\\
E &= \{(x,y,z)|0\le z\le 2,\ z\le y\le 2,\ 0\le x\le (y-z)/2\}\\
E &= \{(x,y,z)|0\le x\le 1,\ 0\le z\le 2-2x,\ z+2x\le y\le 2\}\\
E &= \{(x,y,z)|0\le z\le 2,\ 0\le x\le (2-z)/2,\ z+2x\le y\le 2\}.
\end{aligned}$$

Then

$$\begin{aligned}
\iiint_E f(x,y,z)\,\mathrm{d}V &= \int_0^1\int_{2x}^2\int_0^{y-2x} f(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^2\int_0^{y/2}\int_0^{y-2x} f(x,y,z)\,\mathrm{d}z\mathrm{d}x\mathrm{d}y\\
&= \int_0^2\int_0^y\int_0^{(y-z)/2} f(x,y,z)\,\mathrm{d}x\mathrm{d}z\mathrm{d}y = \int_0^2\int_z^2\int_0^{(y-z)/2} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&= \int_0^1\int_0^{2-2x}\int_{z+2x}^2 f(x,y,z)\,\mathrm{d}y\mathrm{d}z\mathrm{d}x = \int_0^2\int_0^{(2-z)/2}\int_{z+2x}^2 f(x,y,z)\,\mathrm{d}y\mathrm{d}x\mathrm{d}z.
\end{aligned}$$