April 29

Last time:

1. Surface area $= A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$

2. Surface integral of f over $S = \iint_S f \, dS$

$$\boxed{\iint_S f \, dS = \iint_D f(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, dA}$$

3. Surface integral of $\vec{F}$ over $S = \iint_S \vec{F} \cdot d\vec{S}$

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} \, dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dA}$$

where $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$.

"Flux of $\vec{F}$ across S"

In particular, If $\vec{F} = \langle P, Q, R \rangle$ and $S: z = g(x,y)$

then $\vec{r}(x,y) = x\hat{i} + y\hat{j} + g(x,y)\hat{k}$.

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = -g_x\hat{i} - g_y\hat{j} + \hat{k}$$

and $$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(-Pg_x - Qg_y + R\right) dA}$$

Remark: IF a surface is oriented downwards $(\vec{r}_y \times \vec{r}_x)$

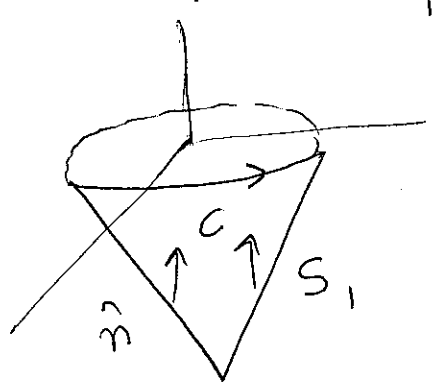
then $$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(Pg_x + Qg_y - R\right) dA}$$

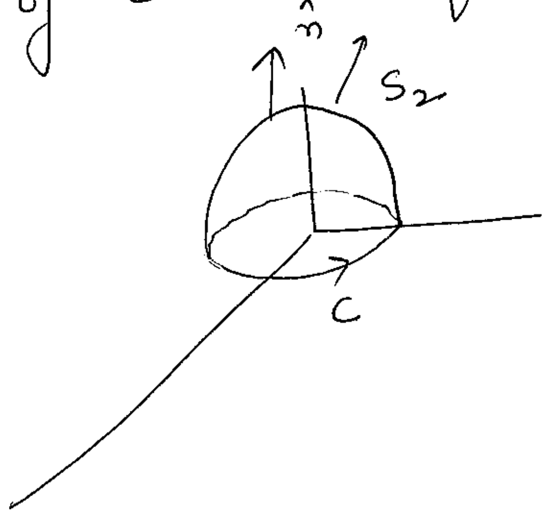
Stokes' theorem

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}(\vec{F})\cdot d\vec{S}}$$

where
- C is a closed curve
- S is any surface bounded by C
- $\vec{F}$ is defined and differentiable everywhere.
- Orientation of C and surface should match up.

Ex1





Ex1 Verify Stokes' theorem for $\vec{F} = z\hat{j} + 2yz\hat{k}$

S is the part of the paraboloid $z = x^2+y^2$ that lies below the plane $z = 1$, Oriented upward.

Sol^n L.H.S: $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 0, 1, +2\sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$

$$= \int_0^{2\pi} \cos t \, dt$$

$$= 0$$

R.H.S: $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$

$$= \iint_{x^2+y^2 \le 1} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z & +2yz \end{vmatrix} \cdot \langle -2x, -2y, 1 \rangle \, dA$$

$$= \iint_{x^2+y^2 \le 1} (+2\underbrace{z}_{x^2+y^2} - 1)(-2x) \, dA$$

$$= \int_0^{2\pi}\int_0^1 (+2r^2 - 1)(-2r^2\cos\theta) \, dr \, d\theta$$

$$= \int_0^1 +(2r^2 - 1)(2r^2) \, dr \int_0^{2\pi} \cos\theta \, d\theta$$

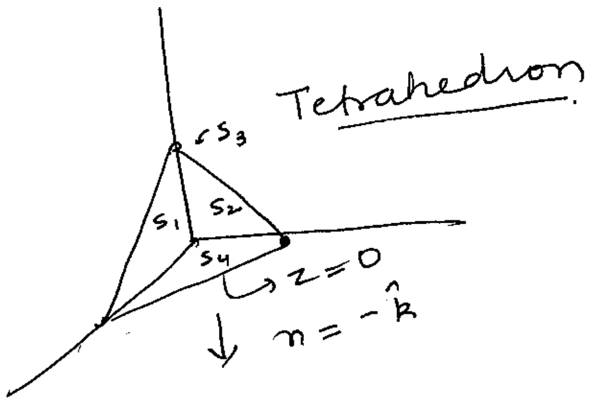
$$= 0.$$

Divergence theorem (Extra for lecture 2)

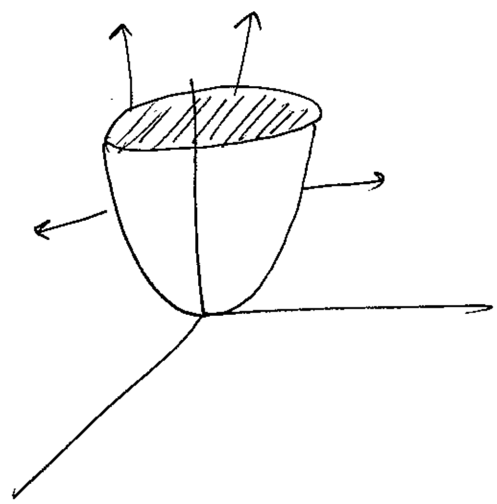
$$\boxed{\oiint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\, dV}$$

- where E is the solid region with closed surface S as the bdry of E
- S is oriented so that normal vector point outwards.

Ex:



Ex:



Ex 2 Verify Divergence theorem for the function $\vec{F} = z\hat{j} + 2yz\hat{k}$ and S consists of the paraboloid $z = x^2+y^2$, $0 \le z \le 1$ and the disk $x^2+y^2 \le 1$.

Sol^n L.H.S:

$$\oiint_S \vec{F}\cdot d\vec{s} = \iint_{S_1} \vec{F}\cdot d\vec{s} + \iint_{S_2} \vec{F}\cdot d\vec{s}$$

On S_1

$$\iint_{S_1} \vec{F}\cdot d\vec{s} = \iint_{x^2+y^2\le 1} -\underset{\substack{\uparrow \\ x^2+y^2}}{z}\, g_y + 2yz\,(1)\; dA$$

$$= \iint_{x^2+y^2\le 1} 0 \; dA = 0$$

~~R.H.S.:~~

On S_2: $z = 1$, $\hat{n} = \langle 0, 0, \hat{k} \rangle$

$$\iint_{S_2} \vec{F}\cdot d\vec{s} = \iint_{x^2+y^2\le 1} 2y \; dA = 0.$$

R.H.S:

$$\iiint_{x^2+y^2\le 1} \int_0^1 2y \; \underset{dz\,dx\,dy}{d\cancel{V}} = \iint_{x^2+y^2\le 1} 2y \; dA = 0.$$