April 27

Last time:

- $\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F}$ and $\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}$
- $\vec{F} = \nabla f \iff \operatorname{curl}(\vec{F}) = 0$
- How to find f? (Learnt in Recitation)
- Vector form of Green's theorem (Learnt in Recitation).

① $$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \hat{T}\, ds = \iint_D \operatorname{curl}(\vec{F}) \cdot \hat{k}\, dA}$$

② $$\boxed{\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_D \operatorname{div}(\vec{F})\, dA}$$

where $\hat{n} = \dfrac{y'(t)}{\sqrt{x'^2(t) + y'^2(t)}}\,\hat{i} - \dfrac{x'(t)}{\sqrt{x'^2 + y'^2}}\,\hat{j}$ is the normal pointing outward.

Aim: To get a version of ① and ② for function of three variables.

Today:

Parametric curves	$\to$ Parametric surfaces
Arc length	$\to$ Surface area
Line Integral	$\to$ Surface integral

Parametric Surfaces

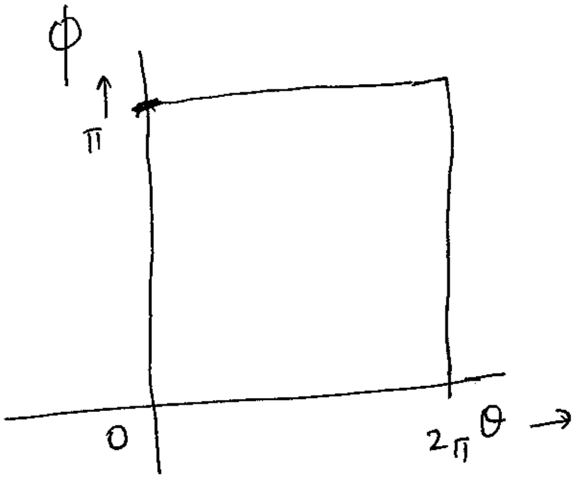
Ex! Sphere of radius 2 : $\rho = 2$

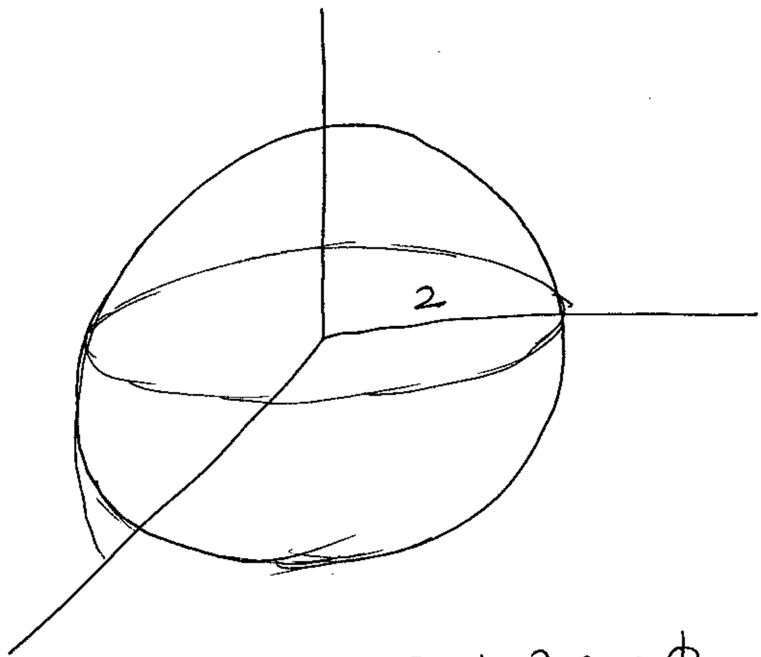
$(x^2+y^2+z^2=4)$

$$x = \rho \sin\phi \cos\theta = 2\sin\phi\cos\theta$$

$$y = \rho \sin\phi \sin\theta = 2\sin\phi \sin\theta$$

$$z = \rho\cos\phi = 2\cos\phi .$$





$$r(\theta,\phi) = 2\sin\phi\cos\theta\,\hat{i} + 2\sin\phi\,\sin\theta + 2\cos\phi .$$

$$0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$$

In particular for $z = f(x,y)$, the formula translates

into $$A(S) = \iint_D |\vec{r}_x \times \vec{r}_y| \, dA$$

$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$

$$= \iint_D |(\hat{i} + f_x\hat{k}) \times (\hat{j} + f_y\hat{k})| \, dA$$

$$(-f_x\hat{i} - f_y\hat{j} \underline{+} \hat{k})$$

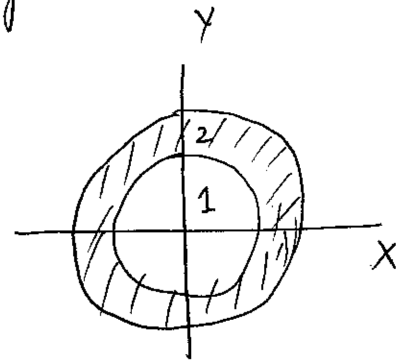
↳ pointing upward

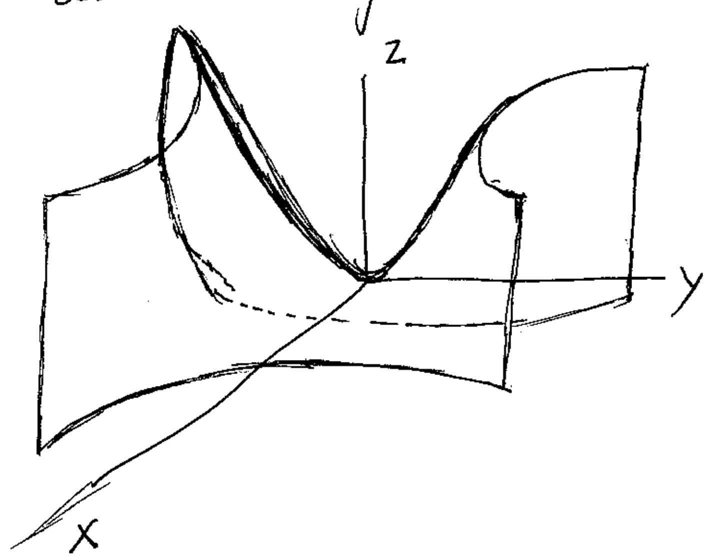
$$\boxed{A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dA}$$

Ex 1 Verify using above definition that surface area of sphere is $4\pi a^2$ where a = radius.

Ex 2 Find the surface area of the part of the surface $z = y^2 - x^2$ that lies within the cylinder $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

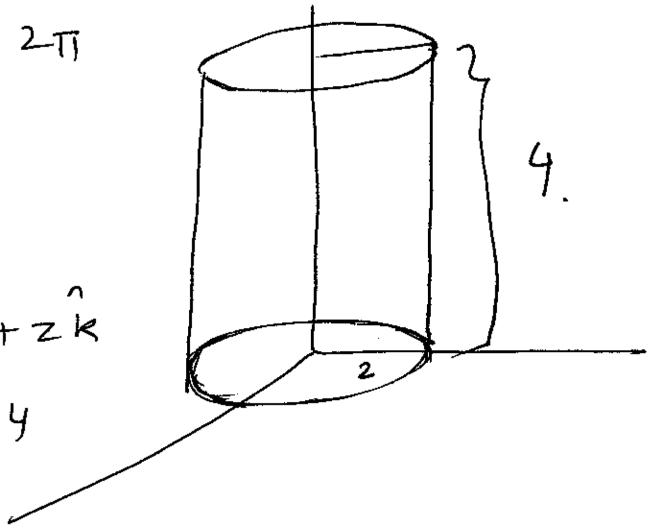
Sol^n





<u>Ex 2</u> <u>Cylinder</u> : $x^2 + y^2 = 4, \quad 0 \le z \le 4$

$x = 2\cos\theta \qquad 0 \le \theta \le 2\pi$
$y = 2\sin\theta$
$0 \le z \le 4$



$r(\theta, z) = 2\cos\theta\,\hat{i} + 2\sin\theta\,\hat{j} + z\,\hat{k}$
$0 \le \theta \le 2\pi, \quad 0 \le z \le 4$

<u>Ex 3</u> <u>Cone</u> : $z = \sqrt{x^2 + y^2}$

$r(x, y) = x\,\hat{i} + y\,\hat{j} + \sqrt{x^2+y^2}\,\hat{k}$

<u>Ex 4</u> $z = f(x, y)$

$r(x, y) = x\,\hat{i} + y\,\hat{j} + f(x, y)\,\hat{k}$

In general, $\vec{r}(u, v) = x(u, v)\,\hat{i} + y(u, v)\,\hat{j} + z(u, v)\,\hat{k}$
(u, v) in D.

$\vec{r}_u = x_u\,\hat{i} + y_u\,\hat{j} + z_u\,\hat{k}, \quad \vec{r}_v = x_v\,\hat{i} + y_v\,\hat{j} + z_v\,\hat{k}$

$\vec{r}_u \times \vec{r}_v$ — normal vector to the tangent plane.

<u>Def</u>: The Surface area of S is given by

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

Sol^n $\vec{r}(x,y) = x\hat{i} + y\hat{j} + (y^2 - x^2)\hat{k}$ where (x,y) belongs to $x^2+y^2=1$ and $x^2+y^2=4$.

$$\text{Area} = \iint_D \sqrt{1 + (-2x)^2 + (2y)^2}\, dA$$

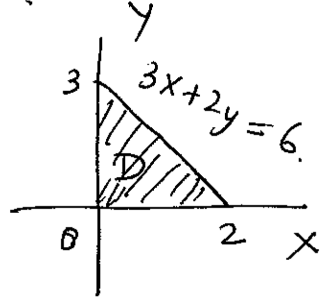
$$= \int_0^{2\pi}\int_1^{2} \sqrt{1+4r^2}\; r\, dr\, d\theta$$

$$= 2\pi \int_{r=1}^{r=2} \sqrt{u}\, \frac{du}{8} \qquad \begin{aligned} u &= 1+4r^2 \\ du &= 8r\,dr \end{aligned}$$

$$= \frac{2\pi}{8} \left. \frac{(1+4r^2)^{3/2}}{3/2} \right]_{r=1}^{r=2}$$

$$= \frac{2\pi}{12}\left[(17)^{3/2} - 5^{3/2}\right] \quad \text{Ans}$$

Ex 3 Find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant $x \geq 0,\ y \geq 0,\ z \geq 0$.



Sol^n $z = 6 - 3x - 2y$

$z \geq 0 \Leftrightarrow \boxed{3x + 2y \leq 6}$

$$\text{Area} = \iint_D \sqrt{1 + (-3)^2 + (-2)^2}\, dA$$

$$= \sqrt{14}\ (\text{area of triangle}) = \sqrt{14}\ \frac{1}{2}(2)(3)$$

$$= 3\sqrt{14} \quad \text{Ans}$$

Surface Integrals (Analogue of line Integrals).

I Given $f(x,y,z)$ and a surface $S: r(u,v) = x\hat{i} + y\hat{j} + z(u,v)\hat{k}$ (u,v) belongs to domain D. Then the surface integral of f over S is defined as

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_D f(r(u,v))\, |r_u \times r_v|\, dA}$$

II Given a vector field $\vec{F}(x,y,z)$ and a surface S described above. Then the surface integral of $\vec{F}$ over S is defined as

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}\, dS}$$

where $\boxed{\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}$ is the unit normal vector.

Another way: $$\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_D (\vec{F}\cdot\hat{n})(\vec{r}(u,v))\, |(\vec{r}_u \times \vec{r}_v)|\, dA$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{(\vec{r}_u \times \vec{r}_v)}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \cancel{|\vec{r}_u \times \vec{r}_v|}\, dA$$

Thus,

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot \hat{n}\, dS = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\, dA}$$

This integral is called "flux of $\vec{F}$ across S"

"Next time"

Another way: IF $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ and

$S : z = g(x,y)$ then

$$r_x \times r_y = -g_x \hat{i} - g_y \hat{j} + \hat{k}$$

(+ indicates upward orientation)

and $$\iint_S \vec{F}\cdot d\vec{S} = \iint_D (-Pg_x - Qg_y + R)\, dA$$

Warning: 1. Orientation of surface is pointing upward!

2. For a surface oriented downward,

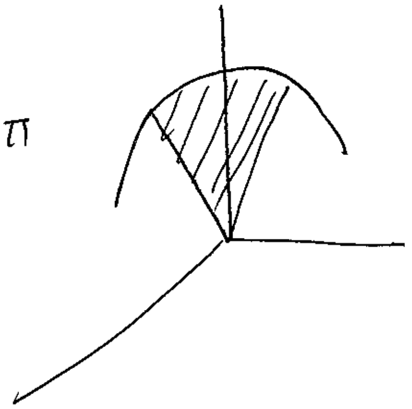
$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_D (-Pg_x - Qg_y + R)\, dA$$

Ex 1 (Extra) Evaluate the surface integral $\iint xyz \, dS$ where S is the part of the sphere $x^2+y^2+z^2=1$ that lies above the cone $z=\sqrt{x^2+y^2}$

Sol^n Here $x = \cos\theta \sin\phi \qquad 0 \le \phi \le \frac{\pi}{4}$

$y = \sin\theta \sin\phi \qquad 0 \le \theta \le 2\pi$

$z = \cos\phi$



$$I = \iint_S xyz \, dS$$

$$= \iint_D \left(\cos\theta \sin\theta \sin^2\phi \cos\phi\right) |\vec{r}_\theta \times \vec{r}_\phi| \, dA$$

where $D = \underset{\phi}{\left[0, \frac{\pi}{4}\right]} \times \underset{\theta}{[0, 2\pi]}$.

$$= \int_0^{2\pi}\int_0^{\pi/4} \left(\cos\theta \sin\theta \sin^2\phi \cos\phi\right) \left| \langle -\sin\theta\sin\phi, \cos\theta, \sin\phi, 0\rangle \times \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle \right| d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \left(\cos\theta \sin\theta \sin^2\phi \cos\phi\right)\left(\sin^2\phi\right) d\phi \, d\theta$$

$$= \left(\int_0^{2\pi} \cos\theta \sin\theta \, d\theta\right)\left(\int_0^{\pi/4} \sin^4\phi \cos\phi \, d\phi\right)$$

$$= \left[\frac{\sin^2\theta}{2}\right]_0^{2\pi} \left[\frac{\sin^5\phi}{5}\right]_0^{\pi/4} =$$ ☺ Ans