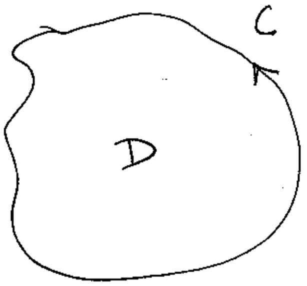
Last time:

Green's theorem: If C is a simple, closed curve oriented counter-clockwise enclosing a region D and if $\vec{F} = P\hat{i} + Q\hat{j}$ where P and Q have continuous first order partial derivatives, then



$$\oint_{C = \underbrace{\partial D}_{\text{boundary of } D}} \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA$$

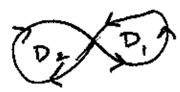
Remarks

1. Green's theorem is a version of F.T.C for functions of two variables.
2. Warning: Only true for closed curves!
3. Pay attention to the orientation of the curve.
   What if C is oriented clockwise?

$$\oint_C \vec{F} \cdot d\vec{r} = -\oint_{-C} \vec{F} \cdot d\vec{r} = -\iint_D (Q_x - P_y)\, dA$$

$$= \iint_D (P_y - Q_x)\, dA$$

4. Non-Simple Curves:

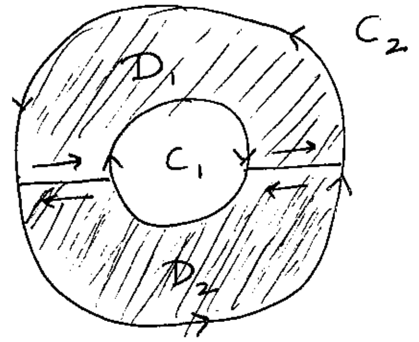


C_2

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (P_y - Q_x)\, dA$$

5. <u>Region with holes</u>:



$$\oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \oint_{\text{Top } C_1} \vec{F}\cdot d\vec{r} + \int_{\text{Bottom } C_1} \vec{F}\cdot d\vec{r} + \int_{\text{Top } C_2} \vec{F}\cdot d\vec{r} + \int_{\text{Bottom } C_2} \vec{F}\cdot d\vec{r}$$

+ 2 white strips

+ 2 white strips

Cancel each other

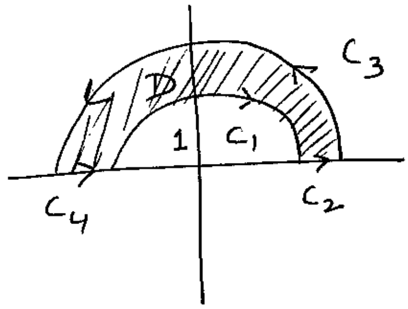
$$= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (P_y - Q_x)\, dA$$

(clockwise)

$$= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (P_y - Q_x)\, dA$$

Verify Green's theorem for the vector function

$$\vec{F}(x,y) = \frac{-y}{x^2+y^2}\,\hat{i} + \frac{x}{x^2+y^2}\,\hat{j}$$



Sol^n To show

$$\iint_D (Q_x - P_y)\, dA = \oint_C \vec{F}\cdot d\vec{r}$$

Note: $Q_x = P_y \Rightarrow$ L.H.S $= \iint_D (Q_x - P_y)\, dA = 0.$

R.H.S: $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$

$$= \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r}.$$

On C_2 and C_4: $y \equiv 0$

$$\Rightarrow P\,dx + Q\,dy \equiv 0$$

$$\Rightarrow \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r} = 0.$$

On C_1: $x(t) = \cos t$ $\quad$ t is from π to 0.
$y(t) = \sin t$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{\pi}^{0} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$$

$$= -\pi$$

On C_3 : $x(t) = 2\cos t$ $\quad 0 \le t \le \pi$

$y(t) = 2\sin t$

$$\int_{C_3} P\,dx + Q\,dy = \int_0^{\pi} \left\langle \frac{-2\sin t}{4}, \frac{2\cos t}{4} \right\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt$$

$$= \int_0^{\pi} \frac{4\sin^2 t + 4\cos^2 t}{4}\, dt$$

$$= \boxed{\pi}$$

$$\Rightarrow \oint_{C} \vec{F}\cdot d\vec{r} = 0 = R.H.S$$

$\Rightarrow$ $L.H.S = R.H.S$

This verfies Green's theorem.

Another application: To find area of regions using line integrals.

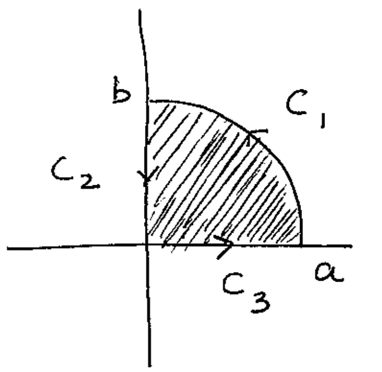
$$\text{Area}(D) = \iint_D 1 \, dA \underset{\uparrow}{=} \oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$

if $Q_x - P_y = 1$

Many choices: $\vec{F} = \underbrace{\langle 0, x \rangle, \langle -y, 0 \rangle}_{\text{works better with non-circular regions}}, \underbrace{\langle -\frac{y}{2}, \frac{x}{2} \rangle}_{\text{better with circular regions}}$

Ex Find the area of the quarter ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Sol^n I $A = \int_0^a \sqrt{\left(1 - \frac{x^2}{a^2}\right)}\, b \, dx$

"Inverse trig sub".

II $A = \oint_C \vec{F} \cdot d\vec{r} = \oint_C -\frac{y}{2} dx + \frac{x}{2} dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$

On C_1: $x(t) = a\cos t$ $\quad 0 \le t \le \frac{\pi}{2}$

$y(t) = b \sin t$

$$\int_0^{\pi/2} \left[\frac{-b \sin t}{2}(-a \sin t) + \frac{a\cos t}{2}(b\cos t)\right] dt = \frac{ab}{2}\left(\frac{\pi}{2}\right) = \frac{\pi ab}{4}$$

On C_2 : $x \equiv 0$

$$-y\,dx + x\,dy \equiv 0$$

On C_3 : $y \equiv 0$

$$-y\,dx + x\,dy \equiv 0$$

$\therefore$ Area $\underset{\uparrow}{=}$ $\frac{\pi}{4}ab$

Green's theorem.

Vector form of Green's theorem

Def: Differential operator : $\underset{\substack{\uparrow \\ \text{vector}}}{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

- If f is a scalar fn then $\nabla f \underset{\substack{\uparrow \\ \text{vector}}}{=}$ gradient of f
- If $\vec{F}$ is a vector fn then we have the following possible operations.

$\nabla \times \vec{F}$

1. $\text{Curl}(\vec{F}) = \nabla \times \vec{F}$

2. Terminology comes from physics.

3. Vector!

4. $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$
$$= (R_y - Q_z)\hat{i} + \hat{j}(P_z - R_x) + (Q_x - P_y)\hat{k}$$

5. $\vec{F}$ is called irrotational if $\text{curl}(\vec{F}) = \vec{0}$

6. In particular, if $\vec{F}(x,y) = \langle P, Q \rangle$ then $\text{curl}(\vec{F}) = (Q_x - P_y)\hat{k}$

$\nabla \cdot \vec{F}$

1. $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$ (divergence of $\vec{F}$)

2. Terminology comes from physics.

3. Scalar!

4. $\text{div}(\vec{F}) = P_x + Q_y + R_y$

5. $\vec{F}$ is called incompressible if $\text{div}(\vec{F}) = 0$

6. In particular, if $\vec{F}(x,y) = \langle P, Q \rangle$ then $\text{div}(\vec{F}) = P_x + Q_y$

Consequences from above

Recall, if $\vec{F}(x,y) = \langle P, Q \rangle$ then $F = \nabla f$ for some f if and only if $Q_x - P_y = 0$ (i.e. $Q_x = P_y$) given that $\vec{F}$ is defined on a certain "nice" region.

1. $\vec{F}(x,y) = \langle P, Q \rangle = \nabla f \iff Q_x = P_y \iff \text{curl}(\vec{F}) = \vec{0}$.

2. We expect a similar result for functions of three variables since curl $(\nabla f) = \vec{0}$ for any f. Indeed we have the following:

Thm: If $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ is defined everywhere, then $\vec{F} = \nabla f \iff \text{curl}(\vec{F}) = \vec{0}$.

Read (Help in Recitation tomorrow).

Now, the question is how to find f once we know that $\vec{F} = \nabla f$?

Ex1 Is $\vec{F} = -ye^{-x}\hat{i} + e^{-x}\hat{j} + 2z\hat{k}$ conservative? If yes then find f so that $\vec{F} = f$

Soln

$$\text{Curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -ye^{-x} & e^{-x} & 2z \end{vmatrix} = 0\hat{i} + 0\hat{j} + (-e^{-x} + e^{-x})\hat{k} = \vec{0}$$

$\Rightarrow \vec{F}$ is conservative, i.e., $\vec{F} = \nabla f$ for some f.

How to find f? Solve three equations:

$$f_x = -ye^{-x} \quad -(1)$$

$$f_y = e^{-x} \quad -(2)$$

$$f_z = 2z \quad -(3)$$

Integrate ① w.r.to x:

$$f(x,y,z) = ye^{-x} + \underbrace{g(y,z)}_{\text{constant with respect to } x.}$$

Note, $f_y(x,y,z) = e^{-x} + g_y(y,z)$
$= e^{-x}$ (from ②)

$\Rightarrow \quad g_y(y,z) = 0$ (Integrate w.r.to y)

$$\Rightarrow \quad g(y,z) = \underbrace{h(z)}_{\substack{\text{constant}\\ \text{with respect}\\ \text{to } y \text{ and of course } x.}}$$

Plug this back in $f(x,y,z)$:

$f(x,y,z) = ye^{-x} + h(z)$.

Note, $f_z(x,y,z) = h'(z)$
$= 2z$ (from ③)

$\Rightarrow \quad h'(z) = 2z \Rightarrow h(z) = z^2 + C$

$$\Rightarrow \quad \boxed{f(x,y,z) = ye^{-x} + z^2 + C}$$

Vector form of Green's theorem

Recall, Green's theorem states that

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA$$

Note, $L\cdot H\cdot S = \oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \left[\frac{d\vec{r}}{ds}\right] ds$

(unit tangent vector)

$$\Rightarrow \quad L\cdot H\cdot S = \oint_C \vec{F} \cdot \hat{T}\, ds \qquad \left(\frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \right)$$

Note, $R\cdot H\cdot S = \iint_D (Q_x - P_y)\, dA$

$$= \iint_D \text{curl}(\vec{F}) \cdot \hat{k}\, dA$$

(vector normal to the plane in which D lies)

New form of Green's theorem:

$$\boxed{\oint_C \vec{F} \cdot \hat{T}\, ds = \iint_D \text{curl}(\vec{F}) \cdot \hat{k}\, dA}$$

This motivates the similar theorem in higher dimension which is called <u>stoke's theorem</u>.

<u>Another form in terms of divergence.</u>

Note, $\iint_D \operatorname{div}(\vec{F})\, dA = \iint_D (\underbrace{P_x + Q_y}_{= P_x - (-Q_y)})\, dA$

$$\overset{G.T}{=} \oint_C \langle -Q, P\rangle \cdot d\vec{r}$$

$$= \oint_C \langle -Q, P\rangle \cdot \left\langle \frac{x'(t)}{\sqrt{x'(t)^2 + (y'(t))^2}}, \frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \right\rangle ds$$

$$= \oint_C \frac{-Q\,x'(t) + P\,y'(t)}{\sqrt{x'^2 + y'^2}}\, ds$$

$$= \oint_C \langle P, Q\rangle \cdot \left\langle \frac{y'(t)}{\sqrt{x'^2 + y'^2}}, \frac{-x'(t)}{\sqrt{x'^2 + y'^2}} \right\rangle ds$$

$$= \oint \vec{F} \cdot \hat{n}\, ds \quad \text{where}$$

$\hat{n} = \left\langle \frac{y'(t)}{\sqrt{x'^2 + y'^2}}, \frac{-x'(t)}{\sqrt{x'^2 + y'^2}} \right\rangle$ is the normal vector pointing outwards.

Thus,

$$\boxed{\oint_C \vec{F} \cdot \hat{n} \, ds = \iint_D \operatorname{div}(\vec{F}) \, dA}$$

This motivates us to write a theorem for functions of three variables which we call <u>divergence theorem</u>.