April 20

t time:

Thm: If $\vec{F}$ is a gradient field ($\vec{F} = \nabla f$) then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{B}) - f(\vec{A}) \quad \text{where}$$

$$\vec{A} = \vec{r}(a) = \text{initial point of } C.$$
$$\vec{B} = \vec{r}(b) = \text{final point of } C.$$

Consequences:

1. If $\vec{F} = \nabla f$ then $\int_C \vec{F} \cdot d\vec{r}$ is "path-independent".
2. If $\vec{F} = \nabla f$ then $\int_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C. ($\vec{A} = \vec{B}$)

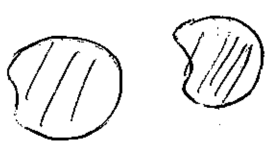
(5 mins)

Diagram:

$$\begin{tikzcd}[column sep=small]
 & \vec{F} = \nabla f \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{path-indendent} \arrow[ur, Rightarrow] \arrow[rr, Leftrightarrow] & & \oint_C \vec{F} \cdot d\vec{r} = 0 \text{ for all } C. \arrow[ul, Rightarrow]
\end{tikzcd}$$

only if $\vec{F}$ is continuous on a region that is open and connected.

Connected:



Not connected

connected

10 mins

Checking whether $\vec{F} = \nabla f$ or not?

Observation: If $\vec{F} = \nabla f$ then $P = f_x$, $Q = f_y$ $\xRightarrow[\text{thm}]{\text{clairaut's}}$ $f_{xy} = f_{yx}$

$P_y = Q_x$

$$\boxed{\vec{F} = \nabla f \Rightarrow P_y = Q_x}$$

Converse: ① If $\vec{F} = \langle P, Q \rangle$ is defined on a open simply connected region

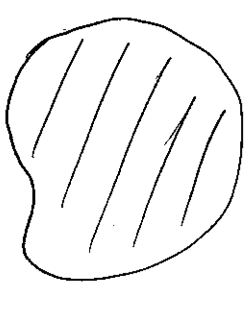
② P, Q have continuous first order partial derivative.

$$\boxed{P_y = Q_x \Longrightarrow \vec{F} = \nabla f}$$

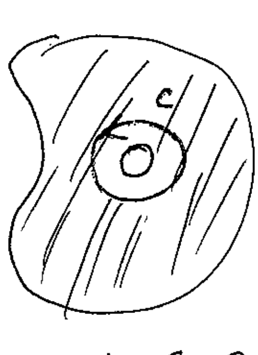
Simply connected: A connected region R is called simply connected if the interior of every simple closed curve is contained in R.

← not simple

← not simple



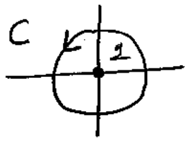
S.C



Not S.C

Eg: $\vec{F}(x,y) = \dfrac{-y}{x^2+y^2}\,\hat{i} + \dfrac{x}{x^2+y^2}\,\hat{j}$

$\boxed{P_y = Q_x}$ but $\vec{F} \neq \nabla f$ because not defined at origin.



$$\oint_C \vec{F} \cdot d\vec{r} = 2\pi$$

Ex 1: $\vec{F} = \langle \underbrace{-y}_{P}, \underbrace{x}_{Q} \rangle \qquad P_y = -1 \neq Q_x = 1$

Not conservative.

Ex 2: $\vec{F} = \langle x^3 + 4xy, \; kx^2\cancel{y} - y^3 \rangle$

Find the value of k so that $\vec{F} = \nabla f$.

Sol^n $\quad P_y = 4x$

$\quad Q_x = 2kx \qquad \Rightarrow \quad k = 2$

Ex 3: $\vec{F} = \langle x^3 + 4xy, \; 2x^2 - y^3 \rangle = \nabla f$

How to find f?

Sol^n Idea is to solve $\boxed{f_x = P \text{ and } f_y = Q}$.

I $\quad f_x = x^3 + 4xy$ (1) and $f_y = 2x^2 - y^3$ (2)

II $\quad f(x,y) = \dfrac{x^4}{4} + 2x^2y + g(y)$

III $f_y = 2x^2 + g'(y) = 2x^2 - y^3$

$$g'(y) = -y^3$$

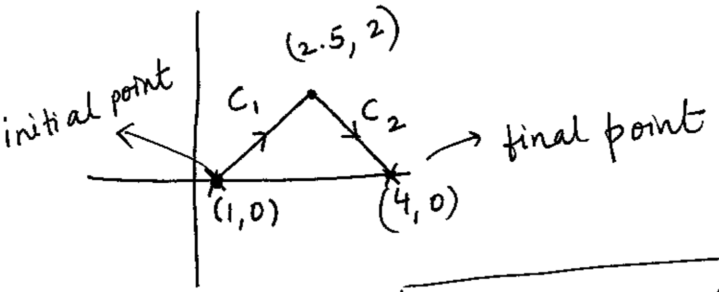
$$\Rightarrow g(y) = -\frac{y^4}{4} + c$$

IV $\boxed{f(x,y) = \frac{x^4}{4} + 2x^2 y + \left(-\frac{y^4}{4}\right) + c}$

Find $\int_C \vec{F}\cdot d\vec{r}$ where C is the triangle drawn below.

Ex4: $\int_C \vec{F}\cdot d\vec{r} = f(4,0) - f(1,0)$

$$= \left(4^3 + c\right) - \left(\frac{1}{4} + c\right) = \boxed{4^3 - \frac{1}{4}}$$



HW 13: When $\vec{F} = \boxed{P\hat{i} + Q\hat{j} + R\hat{k} = \nabla f(x,y,z)\ ?}$ $\boxed{Q26,\ 13.3}$

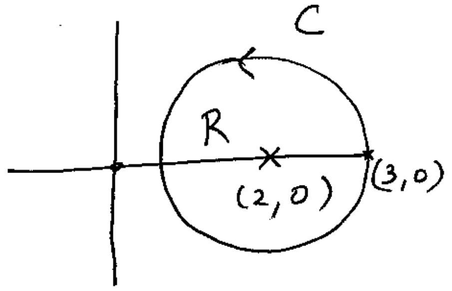
Learn above in HW13.

Green's theorem

If C is a simple closed curve oriented counter-clockwise enclosing a region R. Let $\vec{F} = \langle P, Q \rangle$ where P, Q have continuous first order partial derivatives, then

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_R (Q_x - P_y)\, dA}$$

Ex 1. Compute $\oint_C y e^{-x} dx + (x - e^{-x})\, dy$



Sol^n I C is a closed curve

Is $\vec{F} = \nabla f$? No! $\ddot{\frown}$

$P = y e^{-x}$ $\quad P_y = e^{-x}$

$Q = x - e^{-x}$ $\quad Q_x = 1 + e^{-x}$

$\Rightarrow$ F is not a gradient field.

II ⓐ Direct: $\quad x(\theta) = 2 + \cos\theta \quad 0 \le \theta \le 2\pi$

$y(\theta) = \sin\theta$

⋮

ⓑ Green's thm: $\oint_C \vec{F} \cdot d\vec{r} = \iint_R (1 + e^{-x}) - e^{-x}\, dA$

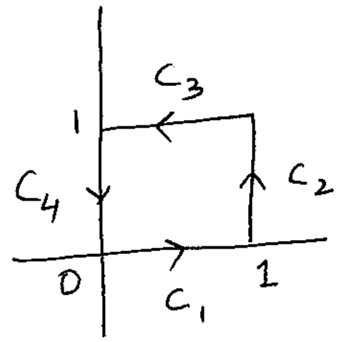
$= \iint_R 1\, dA = \boxed{\pi}$

Ans

Extra Example

Ex 5 Compute $\oint_C x^2 y\,dx + 2xy\,dy$ where C is the bdry of the square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$.

Sol^n Note that C is the joint of four different curves C_1, C_2, C_3, C_4.



Step I: Is $\vec{F} = \nabla f$?

$P = x^2 y$ and $Q = 2xy$

$P_y = x^2 \neq Q_x = 2y$

not equal.

$\Rightarrow$ $\vec{F}$ is not a gradient field! $\ddot{\frown}$

Step II: There are two ways to do the computation: (a) direct computation.

(b) Green's theorem.

(a) **Direct Computation**:

Note: $\oint_C x^2 y\,dx + 2xy\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$

On C_1:

$$\int_{C_1} x^2y\,dx + 2xy\,dy = \int x^2(0)\,dx + 2x(0)0$$
$$= 0$$

(parametrize C_1: $0 \leq x \leq 1$, $y = 0$)

On C_2: parametrize: $0 \leq y \leq 1$ and $x = 1$, $dx = 0$

$$\int_{C_2} x^2y\,dx + 2xy\,dy = \int_{C_2} x^2y\,(0) + 2y\,dy$$
$$= \int_0^1 2y\,dy$$
$$= y^2\Big]_0^1 = 1.$$

On C_3: parametrize: $y = 1$ and x is from 1 to 0. $dy = 0$

(Note: I cannot write $1 \leq x \leq 0$)

$$\int_{C_3} x^2y\,dx + \cancel{2xy\,dy}^{\,0} = \int_1^0 x^2\,dx$$
$$= \frac{x^3}{3}\Big]_1^0 = -\frac{1}{3}$$

On C_4: $x = 0$ and y is from 1 to 0.

$$\int_{C_4} \cancel{x^2y\,dx}^{\,0} + 2x^0y\,dy = \int_{C_4} 0 = 0.$$

Thus $\oint_C x^2y\,dx + 2xy\,dy = 0 + 1 + \left(\frac{-1}{3}\right) + 0$

$= \boxed{\frac{2}{3}}$ Ans

(b) Green's theorem:

$$\oint_C x^2y\,dx + 2xy\,dy = \iint_{\text{Square}} \overset{(Q_x - P_y)}{(2y - x^2)}\,dA$$

$$= \int_0^1\int_0^1 (2y - x^2)\,dx\,dy$$

$$= \int_0^1 \left[2xy - \frac{x^3}{3}\right]_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(2y - \frac{1}{3}\right) dy$$

$$= y^2 - \frac{y}{3}\Big]_0^1 = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

Same Answer!

Note: Green's theorem method turns out to much simpler (shorter)!