Math 21-259

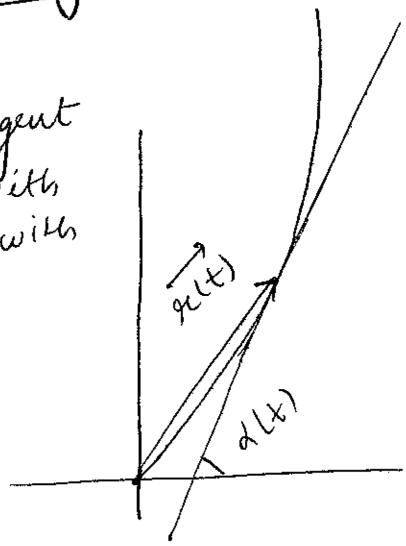
Day 16

Curvature (Continued)

In plane: $\kappa(t) = \left|\dfrac{d\alpha(t)}{ds}\right|$

angle that tangent at the point with p.v $\vec{r}(t)$ makes with x-axis

arc length



In space: Tangent slope $\longleftrightarrow$ unit tangent vector

$$\kappa(t) = \left|\frac{d\hat{T}(t)}{ds}\right|$$

magnitude

$$= \left|\frac{d\hat{T}(t)/dt}{ds/dt}\right|$$

$$\boxed{\begin{aligned}\vec{r}(t) &= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}\\ s(t) &= \int_0^t |\vec{r}'(u)|\,dt\end{aligned}}$$

$$= \left|\frac{\hat{T}'(t)}{s'(t)}\right|$$

$$\boxed{\kappa(t) = \frac{|\hat{T}'(t)|}{|\vec{r}'(t)|}} \qquad \left(\because\ s'(t) = |\vec{r}'(t)|\right)$$

Ex 1 Find the curvature of the straight line.

Sol^n $\vec{r}(t) = (x_0 + ta)\hat{i} + (y_0 + tb)\hat{j} + (z_0 + tc)\hat{k}$

$\vec{r}'(t) = a\hat{i} + b\hat{j} + c\hat{k}$ (direction vector)

$\hat{T}(t) = \dfrac{\langle a, b, c\rangle}{\sqrt{a^2+b^2+c^2}}$, $\hat{T}'(t) = 0 \Rightarrow$ $\boxed{\kappa(t) = 0 \text{ for all } t.}$

Ans

Ex2 Show that for plane curve

$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$, $\boxed{\dfrac{|\hat{T}'(t)|}{|\vec{r}\,'(t)|} = \dfrac{d\alpha(t)}{ds}}$ Ans

Two definitions agree

Ex3 Find the curvature of the circle of radius r.

Sol^n $\vec{r}(t) = r\cos t\,\hat{i} + r\sin t\,\hat{j}$

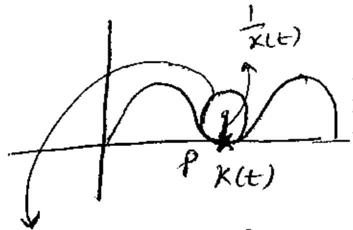
$\boxed{\kappa(t) = \dfrac{1}{r} \text{ for all } t.}$

Curvature is reciprocal of the radius.

Circle of small radius $\longleftrightarrow$ large curvature

Circle of large radius $\longleftrightarrow$ small curvature

In general, Radius of curvature $= \dfrac{1}{\text{curvature}}$



Best describes the curve at P: same tangent and normal $\leftarrow$ $\begin{cases} \text{circle of curvature} \\ = \text{Osculating circle} \end{cases}$

Thm10 (Read pg 573)

$$\boxed{\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}}$$

Useful formula for computation.

In particular for special plane curve, $y = f(x)$.

$\vec{r}(x) = x\hat{i} + f(x)\hat{j}$

$\vec{r}'(x) = \hat{i} + f'(x)\hat{j}, \quad \vec{r}''(x) = f''(x)\hat{j}$

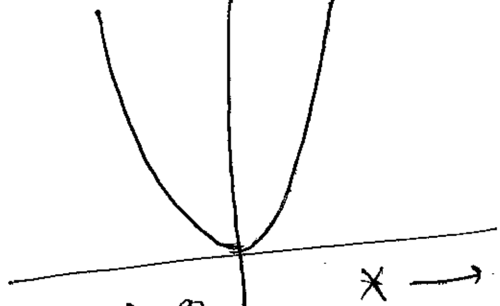
$$\boxed{\kappa(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}}$$

Useful for computation

Ex $y = x^2, \quad f(x) = x^2$

$$\kappa(x) = \frac{2}{\left(1 + (2x)^2\right)^{3/2}}$$

$$\boxed{\kappa(x) = \frac{2}{(1 + 4x^2)^{3/2}}}$$



$x \to \infty$

$\leftarrow x \to -\infty$

becomes flat.

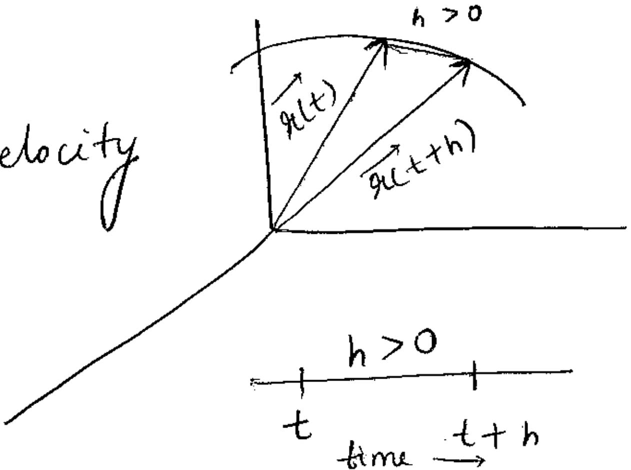
Read this example

As $x \to \pm\infty, \quad \kappa(x) \to 0$

10.9 Applications

$C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ — curve C describes the motion of the particle.

$$\frac{\vec{r}(t+h) - \vec{r}(t)}{h} \to \text{average velocity}$$



Position vector: $\overrightarrow{r(t)} = (x(t))\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$

describes the path of the particle "trajectory".

Velocity vector: $\dfrac{d\vec{r}}{dt} = \vec{v}(t) = x'(t)\,\hat{i} + \dot{y}(t)\,\hat{j} + z'(t)\,\hat{k}$

Speed (Scalar): $|\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

$= \text{function of } t$

You can ask questions such when ~~does~~ the particle have maximum speed or mininimum speed.

(Too many m's) Fix that

Acceleration (vector): $\vec{a} = \dfrac{d\vec{v}}{dt}$

More on this in the video posted on lecture notes page