DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

Math 21-259 Calculus in 3D
Practice Problem from Chapter 13

What sections to expect from Chapter 13 in exam?
Sections 13.1 - 13.8.

In particular, you should be able to answer the following questions.

1. How to Recognize/Sketch Vector Fields?
2. How to check if the given vector field is a gradient field or not? If a gradient field then how to find a potential function?
3. How to compute Line Integrals?
   (a) Directly.
   (b) Using Fundamental Theorem of Line integrals.
   (c) Using Green's theorem (using double integral).
   (d) Stokes' Theorem (using surface integral).
4. Application of Green's Theorem in computing area of bounded regions.
5. Verify Green's theorem.
6. How to compute Surface Area?
7. How to compute Surface Integrals?
   (a) Directly
   (b) Using Stokes' theorem.
8. Verify Stokes Theorem.

Practice Problems including Homework Problems

1. Evaluate the line integral $\int_C (y/x)\,\mathrm{d}s,\ C: x = t^4, y = t^3,\ 1/2 \le t \le 1$. (HW)

2. Evaluate the line integral $\int_C xe^y\,\mathrm{d}x,\ C$ is the arc of the curve $x = e^y$ from (1, 0) to (e, 1).

3. Evaluate the line integral $\int_C \sin x\,\mathrm{d}x + \cos y\,\mathrm{d}y$, C consists of the top half of the circle $x^2 + y^2 = 1$ from (1, 0) to (-1, 0) and the line segments from (-1, 0) to (-2, 3).(HW)

4. Find the work done done by the force field $\mathbf{F}(x, y) = x\sin y\mathbf{i} + y\mathbf{j}$ on a particle that moves along the parabola $y = x^2$ from (-1, 1) to (2, 4). (HW)

5. An object, acted on by various forces, moves along the parabola $y = 3x^2$ from the origin to the point $(1, 3)$. One of the forces acting on the object is $\mathbf{F}(x, y) = x^3\mathbf{i} + y\mathbf{j}$. Calculate the work done by $\mathbf{F}$.

6. Determine the work done by the force $\mathbf{F}(x, y, z) = xy\mathbf{i} + 2\mathbf{j} + 4z\mathbf{k}$ along the circular helix $C : \mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j} + u\mathbf{k}$, from $u = 0$ to $u = 2\pi$.

7. Evaluate the line integral $\int_C \mathbf{h}.\text{d}\mathbf{r}$ if $\mathbf{h}(x, y) = e^y\mathbf{i} - \sin \pi x\mathbf{j}$ and C is the triangle with vertices (1, 0), (0, 1), (-1, 0) traversed counterclockwise.

8. Integrate $\mathbf{h}(x, y, z) = \cos x\mathbf{i} + \sin y\mathbf{j} + yz\mathbf{k}$ over the indicated path:

(a) The line segment from (0, 0, 0) to (2, 3, -1).

(b) $\mathbf{r}(u) = u^2\mathbf{i} - u^3\mathbf{j} + u\mathbf{k}$, u in [0, 1].

9. (a) Determine whether or not $\mathbf{F}(x, y) = (xy\cos xy + \sin xy)\mathbf{i} + (x^2 \cos xy)\mathbf{j}$ is a conservative vector field. If it is, then a function f such that $\mathbf{F} = \nabla f$. (HW)

(b) Compute the line integral of $\mathbf{F}$ over the curve $C : \mathbf{r}(t) = \sqrt{t}\mathbf{i}+(1+t^3)\mathbf{j},\ 0 \le t \le 1$.

(c) Compute the line integral of $\mathbf{F}$ over the curve $C : \mathbf{r}(t) = (t^2 + 3)\mathbf{i} + \sin(\frac{1}{2}\pi t)\mathbf{j}$, $0 \le t \le 1$.

10. (a) Determine whether or not $\mathbf{F}(x, y) = \frac{y^2}{1+x^2}\mathbf{i} + \frac{2y}{\arctan x}\mathbf{j}$ is a conservative vector field. If it is, then a function f such that $\mathbf{F} = \nabla f$. (HW)

(b) Compute the line integral of $\mathbf{F}$ over the curve $C : \mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j},\ 0 \leq t \leq 1$. (HW)

11. Find the work done by the force field $\mathbf{F}(x, y) = e^{-y}\mathbf{i} - xe^{-y}\mathbf{j}$ in moving an object from P(0, 1) to Q(2, 0). (HW)

12. (a) Show that $h(x, y) = (3x^2 + 12xy + 3y^2)\mathbf{i} + (4y^3 + 6x^2 + 6xy)\mathbf{j}$ satisfies the conditions to be a gradient.

(b) Find a function $f(x, y)$ so that $\mathbf{h} = \nabla f$.

(c) Let C be the curve given by $\mathbf{r}(t) = (t + \cos(t))\mathbf{i} + (\sin(t) + \cos^3(t))\mathbf{j}$, $0 \leq t \leq \pi/2$, and compute

$$\int_C \mathbf{h}.\mathrm{d}\mathbf{r}.$$

13. Compute $\int_C \mathbf{G}.\mathrm{d}\mathbf{r}$, where $\mathbf{G}(x, y) = (2xy + e^x - 3)\mathbf{i} + (x^2 - y^2 + \sin y)\mathbf{j}$ and C is the ellipse $4x^2 + 9y^2 = 36$.

14. Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem. (HW)

   (a) $\oint_C xy^2\,\mathrm{d}x + x^3\,\mathrm{d}y$, C is the rectangle with vertices (0, 0), (2, 0), (2, 3), and (0, 3).

   (b) $\oint_C x\,\mathrm{d}x + y\,\mathrm{d}y$, C consists of the line segments from (0, 1) to (0, 0) and from (0, 0) to (1, 0), and the parabola $y = 1 - x^2$ from (1, 0) to (0, 1).

15. Use Green's theorem to evaluate the following line integrals:

(a) $\int_C xe^{-2x}\,\mathrm{d}x + (x^4 + 2x^2y^2)\,\mathrm{d}y$, C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. (HW)

(b) $\int_C y^2 \cos x\,\mathrm{d}x + (x^2 + 2y\sin x)\,\mathrm{d}y$, C is the triangle from (0, 0) to (2, 6) to (2, 0) to (0, 0). (HW)

16. Use Green's theorem to evaluate $\int_C y^2 \, \mathrm{d}x + (3x + 2xy) \, \mathrm{d}y$ where C is the circle of radius 2 oriented counterclockwise.

17. Compute the integral of the function $\mathbf{F}(x, y) = (e^{x^2} + 2x^2 - 4x + y)\mathbf{i} + (\sin y + x + 3y + 2)\mathbf{j}$ over the curve $C : r(\theta) = \sin\theta, \ 0 \leq \theta \leq \pi$.

18. Let C be the circle of radius 2 centered at $(2, 3)$ oriented counterclockwise.

(a) Find a parameterization of C using $0 \leq t \leq 2\pi$.

(b) Use your parameterization of C to express

$$\int_C (4xy + 2y)\,\mathrm{d}x + (2x^2)\,\mathrm{d}y$$

as an integral $\mathrm{d}t$.

(c) Use Green's theorem to compute this integral.

19. Use Green's theorem to find the area of the following bounded regions.

(a) Region bounded by the astroid $\mathbf{r}u = \cos^3 u\mathbf{i} + \sin^3 u\mathbf{j},\ 0 \leq u \leq 2\pi$.

(b) Region in the second quadrant bounded by the ellipse $4x^2 + 9y^2 = 36$.

20. (a) Use Green's theorem to show that if is the region enclosed by a simple closed curve C, then $\oint_C (x + 2y)\,\mathrm{d}x + (3x - 4y)\,\mathrm{d}y = area(\Omega)$.

(b) What is the value of the integral if

i. C is the circle: $(x - 1)^2 + (y - 2)^2 = 4$ and

ii. C is the path formed by joining the four points: A(-1, 0), B(1, 0), C(1, 1), D(-1, 1).

iii. C is the path formed by the y-axis, the line y = 1, and the line $y = \frac{1}{2}x$.

21. Verify Green's theorem for the vector field $\mathbf{F}(x, y) = (1+10xy+y^2)\,\mathrm{d}x+(6xy+5x^2)\,\mathrm{d}y$ and the curve C which is the the square with vertices (0, 0), (a, 0), (a, a), (0, a).

22. Find the curl and the divergence of the vector field

$$\mathbf{F}(x, y) = \frac{x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k}.$$

(HW)

23. Determine whether or not the vector field $\mathbf{F} = e^z\mathbf{i} + \mathbf{j} + xe^z\mathbf{k}$ is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$. (HW)

24. Find the surface area of the part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$. (HW 15)

25. Find the surface area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices (0, 0), (0, 1), and (2, 1).(HW 15)

26. Find the surface area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.(HW 15 – example done in class)

27. Evaluate the surface integral $\iint_S \sqrt{1 + y^2 + z^2}\, \mathrm{d}S$, where S is the helicoid with vector equation $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k},\ 0 \le u \le 1,\ 0 \le v \le \pi$. (HW 15)

28. Evaluate the surface integral $\iint_S xy\,\mathrm{d}S$, where S is the triangular region with vertices (1, 0, 0), (0, 2, 0), and (0, 0, 2).(HW 15)

29. Evaluate the surface integral $\iint_S yz\,\mathrm{d}S$, where S is the part of the plane $x + y + z = 1$ that lies in the first octant.(HW 15)

30. Evaluate the surface integral $\iint_S xyz\,\mathrm{d}S$, where S is the part of the sphere $x^2+y^2+z^2 = 1$ that lies above the cone $z = \sqrt{x^2+y^2}$. (HW 15)

31. Evaluate the surface integral $\iint_S \mathbf{F}.\mathrm{d}\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant, with orientation towards the origin.(HW 15)

32. Evaluate the surface integral $\iint_S \mathbf{F}.\mathrm{d}\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\mathbf{j} - z\mathbf{k}$ and S consists of the paraboloid $y = x^2 + z^2,\ 0 \leq y \leq 1$, and the disk $x^2 + z^2 \leq 1,\ y = 1$.(HW 15)

33. Evaluate the surface integral $\iint_S \mathbf{F}.d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\mathbf{i} + (z - y)\mathbf{j} + x\mathbf{k}$ and S is the surface of the tetrahedron with vertices (0, 0, 0), (1, 0, 0), and (0, 0, 1).(HW 15)

34. Use Stokes' theorem, that is, line integrals to evaluate $\iint_S \mathbf{F}.d\mathbf{S}$ where

(a) $\mathbf{F}(x, y, z) = x^2e^{yz}\mathbf{i} + y^2e^{xz}\mathbf{j} + z^2e^{xy}\mathbf{k}$ and S is the hemisphere $x^2 + y^2 + z^2 = 4,\ z \geq 0$,oriented upwards.(HW 15)

(b) $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward.(HW 15)

35. Use Stokes' theorem, that is, surface integrals to evaluate $\int_C \text{curl}\mathbf{F}.d\mathbf{r}$ where

(a) $\mathbf{F}(x, y, z) = e^{-x}\mathbf{i} + e^{x}\mathbf{j} + e^{z}\mathbf{k}$, and C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant oriented counterclockwise as viewed from above.(HW 15)

(b) $\mathbf{F}(x, y, z) = xy\mathbf{i}+2z\mathbf{j}+3y\mathbf{k}$, and C is the curve of intersection of the plane $x+z = 5$ and the cylinder $x^2 + y^2 = 9$.(HW 15)

36. Verify Stokes' theorem for the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + xyz\mathbf{k}$ and S is the part of the plane $2x + y + z = 2$ that lies in the first octant oriented upward.(HW 15)