Problem set on Cross Product MM

1. Calculate the vector product of **a** and **b** given that

 a= 2**i** + **j** + **k** and **b** = **i** – **j** – **k** (Ans. 3 **j** - 3 **k**)

2. Calculate the vector product of **i** - **j** and **i** + **j**. (Ans. 2 **k**)

3. Find the unit vectors that are perpendicular to both **i**+2**j**+**k**

 and 3**i**-4**j**+2**k**. (**Ans.** $\pm \frac{1}{\sqrt{165}} (8\mathbf{i} + \mathbf{j} - 10\,\mathbf{k})$

4. Find a vector **N** that is perpendicular to the plane determined

 by the points P(1, 2, 3), Q(-1, 3, 2), R(3, -1, 2), and find the area

 of the triangle. (Ans. $-4\mathbf{i} - 4\mathbf{j} + 4\,\mathbf{k}$ and $\mathbf{area} = 2\sqrt{3}$)

5. Find the determinant of the vectors

 a = 2**i** + 3**j** + **k**, **b** = 3**i** + 5**j** – **k**, and **c** = **i** – **j**

  (Ans. – 13)

6. Using the identity **a** $\times$ (**b** $\times$ **c**) = (**a** . **c**)**b** - (**a** . **b**)**c**,

 calculate (i + j)$\times$ [(i - j) $\times$(i + k)]. (Ans. $\mathbf{i} - \mathbf{j}$)

7. Given the points O(0, 0, 0), P(1, 2, 3), Q(1, 1, 2), R(2, 1,

 1), find the volume of the parallelepiped with edges $\overrightarrow{OP}$,

 $\overrightarrow{OQ}$, and $\overrightarrow{OR}$. (Ans. Vol = 0)

 The above answer can be explained by noting that the given

 vectors are coplanar which implies that height of the

 parallelepiped is zero. In other words, no parallelepiped is

 formed.

8. Calculate [(**i** + **j**)$\times$ (**i** - **j**)] . (**k** - **i**) (Ans. - 2)

9. Calculate [(**i** - **j**) $\times$ (**k** - **i**)] . (**i** + **j**) (Ans. - 2)

10. Check whether the following vectors are coplanar or not.

   u = **i**+5**j** − 2**k**, v = 3**i** − **j**, and w = 5**i** + 9**j** − 4**k**

   (Ans. Coplanar)

11. The points P (1, 2, 3); Q(1, 3, 2); R(3, 1, 2) determine a plane P. Find a vector **N** that is perpendicular to P.

   (Ans. $- 2\mathbf{i} - 2\mathbf{j} - 2\,\mathbf{k}$)

12. Find two unit vectors orthogonal to both <2, 0, -3> and

   <-1, 4, 2>.   (Ans. $\pm \frac{1}{\sqrt{209}} < 12, -1,\ 8 >$)

13. Given two vectors in space $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3)$.

   - Show that the area of the parallelogram formed by vectors **a** and **b** is given by $|a|^2 |b|^2 - (a.b)^2$.
   - Further , show that
     $$|a|^2|b|^2 - (a.b)^2 = (a_1 b_2 - b_2 a_1)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_3 b_2 - a_2 b_3)^2$$
   - Finally, conclude that the area of the parallelogram formed by vectors **a** and **b** is same as $|a \times b|$.

14. Mark True or False. Give a proper reasoning if true and a counter example if false.

   - Dot Product of two unit vectors is again a unit vector.

     False; dot product is a scalar
   - Cross Product of two unit vectors is again a unit vector.
     False; the length of the cross product of two unit vectors is equal to sine of the angle between them which will be equal to one only if the angle is 90 degrees.

- Dot Product of a vector with itself is equal to the square of its length. True; this follows easily by the definition.
- Cross Product of a vector with itself is equal to the square of the same vector.　False; cross Product of a vector with itself is a zero vector.
- Cross product of two vectors **a** and **b** is equal to the determinant of the vector **a** and **b**.
  False; If we assume that the vectors **a** and **b** are vectors in space then the determinant does not makes sense and if we assume that **a** and **b** are two vectors in plane then the cross product does not makes sense.
- For any two vectors **u** and **v**, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.
  True; Both represent the area of a parallelogram formed by vector **u** and **v**.
- For any two vectors **u** and **v**, $(\mathbf{u} \times \mathbf{v}).\,\mathbf{u} = \mathbf{0}$.
  True; by definition $\mathbf{u} \times \mathbf{v}$ is a vector that is perpendicular to both **u** and **v**.
- For any three vectors **u**, **v** and **w**,
  $$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$
  False; If we take u = i − j, v = j, w = k then that gives us a counter example to above.
- For any three vectors **u**, **v** and **w**,
  $$(\mathbf{u} \times \mathbf{v}).\,\mathbf{w} = \mathbf{u} \times (\mathbf{v}.\,\mathbf{w})$$
  False; the right hand side does not even make any sense.
- For any three vectors **u**, **v** and **w**,
  $$(\mathbf{u} \times \mathbf{v}).\,\mathbf{w} = \mathbf{u}\,.\,(\mathbf{v} \times \mathbf{w}) = (\mathbf{w} \times \mathbf{u}).\,\mathbf{v}$$
  True; the above transition is done in a cyclic order.

- If $\mathbf{u} \cdot \mathbf{v} = 0,$ then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
  False; dot product can also be zero when the vectors are perpendicular.
- If the vector $\mathbf{u}$ is parallel to the vector $\mathbf{v}$ then
$$\mathbf{u} \times \mathbf{v} = \mathbf{0}.$$
  True; If the vectors are parallel then the angle between them is zero or π which further implies that **|u × v| = 0.**
- If the vector $\mathbf{u}$ is parallel to the vector $\mathbf{v}$ then
$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|.$$
  False; Since the angle between the parallel vectors can either be zero or π which means $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- If the vector $\mathbf{u}$ is perpendicular to the vector $\mathbf{v}$ then
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|.$$
  True; $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta = |\mathbf{u}||\mathbf{v}|\sin\frac{\pi}{2} = |\mathbf{u}||\mathbf{v}|$.
- If the vector $\mathbf{u}$ is perpendicular to the vector $\mathbf{v}$ then
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{0}.$$
  True; By geometrical definition of the dot product.
- If the direction of the vector $\mathbf{u}$ is same as the direction of the vector **v, then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$.**
  False; the angle between u and v is equal to zero.
- If a, b, and **c** are three non-zero vectors, such that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ then it is true that **b = c.**
  False; not necessarily because $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ implies that $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$ which can happen when the vector a is parallel to the vector (**b - c**) without forcing any vector be zero.

- If a, b, and **c** are three non-zero vectors, such that
  a . **b** = **a** . **c** then it is true that **b = c.**
  False; similar reasoning hold true even in this case. It can happen when the vector a is perpendicular to the vector (b - c).

- If a, b, and **c** are three non zero vectors, such that
  a × **b** = **a** × **c** and **a** . **b** = **a** . **c** then it is true that
  b = c.
  True; **a** × **b** = **a** × **c** implies that the vector **a** is perpendicular to the vector **(b - c)** and **a** . **b** = **a** . **c** implies that the vector a is parallel to the vector **(b - c)** which is not possible unless one of **a** or **(b - c)** equals zero. Since the vector **a** is given to be non-zero, therefore **b - c = 0** which further implies that **b = c.**

15. If a and b are vectors, and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ and $\mathbf{a} . \mathbf{b} = 0$; then
   (a). $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$.
   (b). $\mathbf{a} = \mathbf{0}$.
   (c). At least one of **a** and **b** is **0**.
   (d). $\mathbf{b} = \mathbf{0}$.
   (e). None of the above.

16. If $\mathbf{a} \times \mathbf{i} = \mathbf{0}$ and $\mathbf{a} \times \mathbf{j} = \mathbf{0}$ then **a**
   (a). is perpendicular to $\mathbf{i} \times \mathbf{j}$
   (b). is parallel to $\mathbf{i} \times \mathbf{j}$
   (c). is equal to $\mathbf{k}$
   (d). is equal to **0**
   (e). cannot be determined

17. If a, b, and **c** are three non zero vectors, such that
$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$; then
(a). $\mathbf{b} = \mathbf{c}$
(b). **a** is parallel to the vector $(\mathbf{b} - \mathbf{c})$.
(c). **a** is perpendicular to the vector $(\mathbf{b} - \mathbf{c})$.
(d). Nothing can be said.
(e). None of the above.

18. If a, b, and **c** are three non zero vectors, such that
$\mathbf{a} . \mathbf{b} = \mathbf{a} . \mathbf{c}$; then
(a). $\mathbf{b} = \mathbf{c}$
(b). **a** is parallel to the vector $(\mathbf{b} - \mathbf{c})$.
(c). **a** is perpendicular to the vector $(\mathbf{b} - \mathbf{c})$.
(d). Nothing can be said.
(e). None of the above.

19. If a, b, and **c** are three non zero vectors, such that
$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ and $\mathbf{a} . \mathbf{b} = \mathbf{a} . \mathbf{c}$; then
(a). $\mathbf{b} = \mathbf{c}$
(b). **a** is parallel to the vector $(\mathbf{b} - \mathbf{c})$.
(c). **a** is perpendicular to the vector $(\mathbf{b} - \mathbf{c})$. .
(d). Nothing can be said.
(e). None of the above.