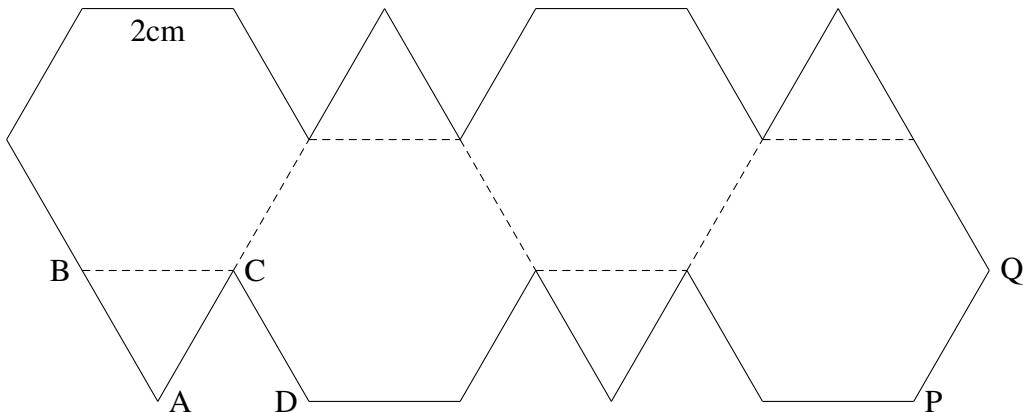


## 37th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 24, 2015

### Fill out the individual registration form

1. Find all integers  $n$  for which  $n^4 + 6n^3 + 11n^2 + 3n + 31$  is a perfect square.
2. The planar diagram below, with equilateral triangles and regular hexagons, sides length 2cm., is folded along the dashed edges of the polygons, to create a closed surface in three dimensional Euclidean spaces. Edges on the periphery of the planar diagram are identified (or glued) with precisely one other edge on the periphery in a natural way. Thus for example, BA will be joined to QP and AC will be joined to DC. Find the volume of the three-dimensional region enclosed by the resulting surface.



3. Let  $(a_i)_{1 \leq i \leq 2015}$  be a sequence consisting of 2015 integers, and let  $(k_i)_{1 \leq i \leq 2015}$  be a sequence of 2015 positive integers (positive integer excludes 0). Let

$$A = \begin{pmatrix} a_1^{k_1} & a_1^{k_2} & \cdots & a_1^{k_{2015}} \\ a_2^{k_1} & a_2^{k_2} & \cdots & a_2^{k_{2015}} \\ \vdots & \vdots & \cdots & \vdots \\ a_{2015}^{k_1} & a_{2015}^{k_2} & \cdots & a_{2015}^{k_{2015}} \end{pmatrix}.$$

Prove that  $2015!$  divides  $\det A$ .

(Please turn over)

4. Consider the harmonic series  $\sum_{n \geq 1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ . Prove that every positive rational number can be obtained as an *unordered* partial sum of this series. (An unordered partial sum may skip some of the terms  $\frac{1}{k}$ .)

5. Evaluate  $\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx$  (where  $0 \leq \arctan(x) < \pi/2$  for  $0 \leq x < \infty$ ).

6. Let  $(a_1, b_1), \dots, (a_n, b_n)$  be  $n$  points in  $\mathbb{R}^2$  (where  $\mathbb{R}$  denotes the real numbers), and let  $\varepsilon > 0$  be a positive number. Can we find a real-valued function  $f(x, y)$  that satisfies the following three conditions?

- $f(0, 0) = 1$ ;
- $f(x, y) \neq 0$  for only finitely many  $(x, y) \in \mathbb{R}^2$ ;
- $\sum_{r=1}^{r=n} |f(x + a_r, y + b_r) - f(x, y)| < \varepsilon$  for every  $(x, y) \in \mathbb{R}^2$ .

Justify your answer.

7. Let  $n$  be a positive integer and let  $x_1, \dots, x_n$  be  $n$  nonzero points in  $\mathbb{R}^2$ . Suppose  $\langle x_i, x_j \rangle$  (scalar or dot product) is a rational number for all  $i, j$  ( $1 \leq i, j \leq n$ ). Let  $S$  denote all points of  $\mathbb{R}^2$  of the form  $\sum_{i=1}^{i=n} a_i x_i$  where the  $a_i$  are integers. A closed disk of radius  $R$  and center  $P$  is the set of points at distance at most  $R$  from  $P$  (includes the points distance  $R$  from  $P$ ). Prove that there exists a positive number  $R$  and closed disks  $D_1, D_2, \dots$  of radius  $R$  such that

- Each disk contains exactly two points of  $S$ ;
- Every point of  $S$  lies in at least one disk;
- Two distinct disks intersect in at most one point.