

TRIPLES

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VALID SEQUENCE

(1, 1, 1)

(1, 2, 3)

(1, 4, 4)

(2, 5, 1) *for every two rows, at least two coordinates increase*

(3, 1, 5)

(1, 6, 6)

(2, 7, 8)

PUZZLE

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QUESTION

What's the length of longest valid sequence from $\{1, \dots, L\}$?

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QUESTION

What's the length of longest valid sequence from $\{1, \dots, L\}$?

OBSERVATION

The length is at most L^2 .

THEOREM (ERDŐS-SZEKERES 1935)

Every permutation of $\{1, \dots, n\}$ has a monotone subsequence of length at least \sqrt{n} .

MONOTONE SEQUENCES

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EXAMPLE

1 5 2 7 3 6 4

MONOTONE SEQUENCES

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Every permutation of $\{1, \dots, n\}$ has a monotone subsequence of length at least \sqrt{n} .

EXAMPLE

1 5 2 7 3 6 4

Proof. Under each number, write lengths of longest increasing and decreasing subsequences ending there.

	1	5	2	7	3	6	4
inc.	1	2	2	3	3	4	4
dec.	1	1	2	1	2	2	3

PROPOSITION

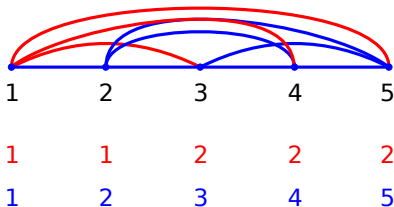
Every 2-coloring of edges of $K_{\{1,\dots,n\}}$ has a monochromatic forward path of length at least \sqrt{n} .

ORDERED RAMSEY THEORY

PROPOSITION

Every 2-coloring of edges of $K_{\{1,\dots,n\}}$ has a monochromatic forward path of length at least \sqrt{n} .

Proof. Under each vertex, write lengths of longest red/blue paths ending there.

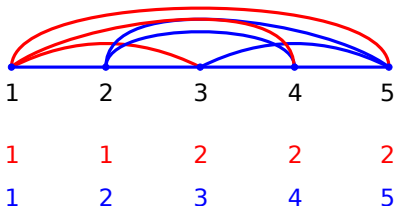


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Tight:



PROPOSITION

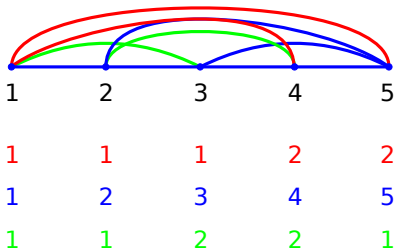
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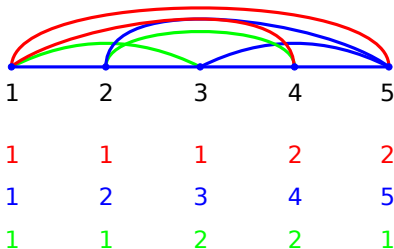


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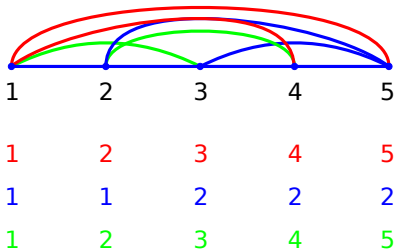
Every 3-coloring of edges of $K_{\{1,\dots,n\}}$ has a non-rainbow forward path of length at least $\sqrt[3]{n}$.

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Every 3-coloring of edges of $K_{\{1,\dots,n\}}$ has a non-rainbow forward path of length at least $\sqrt[3]{n}$.

Proof. Under each number, write lengths of longest red-free/blue-free/green-free paths ending there.

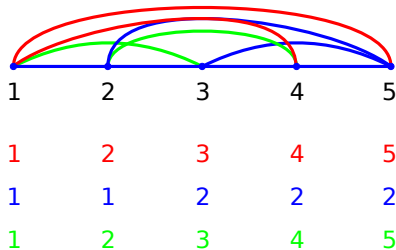


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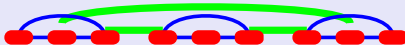
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PROPOSITION

There is a 3-coloring of edges of $K_{\{1,\dots,n\}}$ where all non-rainbow forward paths have length at most $n^{2/3}$.

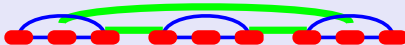


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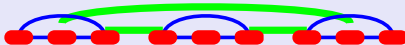
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COROLLARY

There is a valid sequence of triples of length at least $L^{3/2}$.

OBSERVATION

Every valid sequence of triples has length at most L^2 .

DEFINITION

Given n vertices, and $2k - 1$ preference orderings on them (permutations of $1, \dots, n$), the k -majority tournament has \overrightarrow{ij} when majority of orderings prefer i over j .

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APPLICATION TO RAMSEY

Potential source of constructions: control size of largest transitive subtournament.

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There is a 2-coloring of the edges of K_n where all monochromatic cliques have order at most $2 \log_2 n$.

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CHALLENGE

Discover interesting new Ramsey constructions, esp. explicit.

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Preference orderings. $2k - 1$ in total

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- ② Lexicographic, coordinates prioritized $2, 3, \dots, r, 1$.
- ③ Lexicographic, coordinates prioritized $3, 4, \dots, r, 1, 2$.
- ④ etc.

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Other orderings of less-significant coordinates should be used.

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SPECIAL TRANSITIVE SUBTOURNAMENTS

If no tiebreaks necessary, for every two tuples, the later one is greater in at least half the coordinates.

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THEOREM (L.)

There is a valid sequence of triples of length at most $L^2 / \log^* L$.

DEFINITION

Tower function $T(n) = 2^{2^{\cdot^{\cdot^{\cdot^2}}}}$. Inverse function is $\log^* n$.

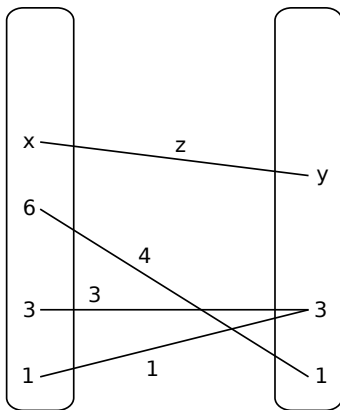
AUXILIARY GRAPH

(1, 3, 1)

(3, 3, 3)

(6, 1, 4)

(x, y, z)



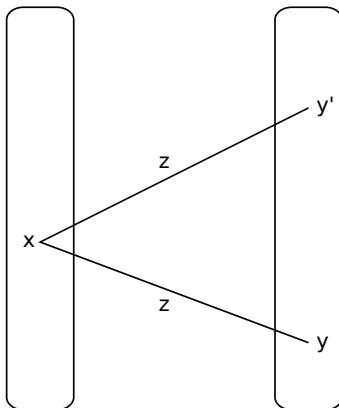
MATCHING

OBSERVATION

Same-labeled edges form a matching.

Proof.

(x, y, z)
...
 (x, y', z)



NON-CROSSING MATCHING

OBSERVATION

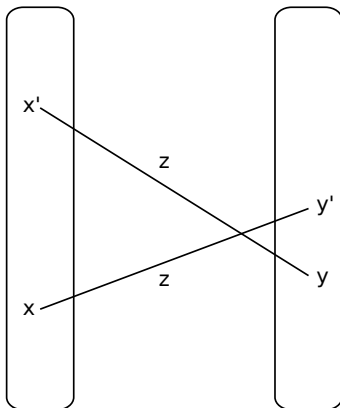
Same-labeled edges form a non-crossing matching.

Proof.

(x, y', z)

...

(x', y, z)



INDUCED MATCHING

OBSERVATION

Same-labeled edges form an induced matching.

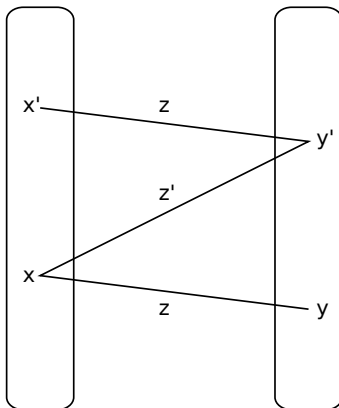
Proof.

(x, y, z)

...

(x, y', z')

$z < z' < z$



QUESTION

Given a graph with $2n$ vertices, where edges can be decomposed into n induced matchings. What is maximum possible number of edges?

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Maximum is $n^2 / \log^* n$ edges.

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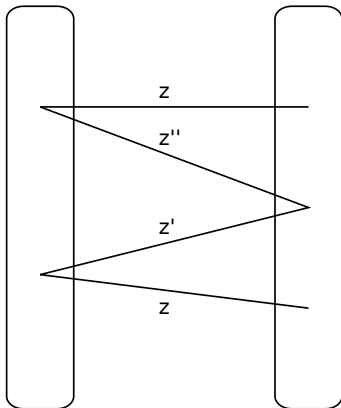
It is possible to achieve $n^2 / e^{\sqrt{\log n}}$ edges.

OBSERVATION

Same-labeled edges form a Σ -free matching.

Proof.

$$z < z' < z'' < z$$



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PROPOSITION

If also forbid “floppy” Σ , max number of edges is at most $n^{3/2}$.

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Any improved bound transfers to triples problem.