Matching Number Process

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Joint work with Michael Krivelevich and Benny Sudakov

Erdős-Rényi Random Graph Process

- Start: n vertices, no edges
- Add $\binom{n}{2}$ edges in uniformly random permutation

Random Graph Processes

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Theorem (Erdős-Rényi)

At cn edges:

- If $c < \frac{1}{2}$, largest component has order $\log n$ a.a.s.
- If $c > \frac{1}{2}$, largest component has order n a.a.s.

Theorem (Bollobás)

Hamilton cycle appears the moment min degree ≥ 2 a.a.s.

Erdős

There are n-vertex graphs with all cliques and indep sets $\leq 2 \log n$.

Applications

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Triangle-Free Process

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Problem (Bollobás-Erdős)

Analyze Triangle-Free Process for off-diagonal Ramsey construction

Results

- Erdős, Suen, Winkler: First Triangle-Free Process analysis, to $n^{3/2}$ edges

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Restricted Random Processes

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- Bohman, Keevash: H-free process for arbitrary fixed H
- Bohman, Frieze, Lubetzky: Triangle-removal process

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Define $\phi = \mathbb{E}_v[\text{size of component containing } v] = \frac{1}{n} \sum |C_i|^2$, where C_i are the connected components.

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Observation

- C_i and C_j merge: $C_i^2 + C_j^2 \longrightarrow (C_i + C_j)^2 \quad \ldots \quad +2C_iC_j$.

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OBSERVATION

- C_i and C_j merge: $C_i^2 + C_j^2 \longrightarrow (C_i + C_j)^2 \quad \ldots \quad +2C_iC_j$.
- Difference equation $\Delta\phi = \frac{1}{n}2\phi^2$

$$\frac{\phi'}{\phi^2} = \frac{2}{n}$$

$$-\frac{1}{\phi} = \frac{2}{n}t + C$$

Differential Equations

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- At $t = 0$, $\phi = 1$, so $C = -1$.

$$\phi = \frac{1}{1 - \frac{2}{n}t}$$

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- Used Wormald's Differential Equations Method
- Obstructions had bounded size

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Unbounded obstruction sizes?

Analyze random graph process, with matching number $\leq k = k(n)$.

Definition

Matching number is maximum number of vertex-disjoint edges

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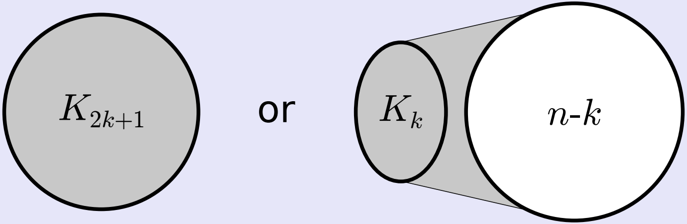
Observation

If $k = 1$, process always results in star $K_{1,n-1}$

THEOREM (ERDŐS-GALLAI)

The *n*-vertex graphs with matching number $\leq k$ which maximize the number of edges are:



Extremal Results

Theorem (Erdős-Gallai)

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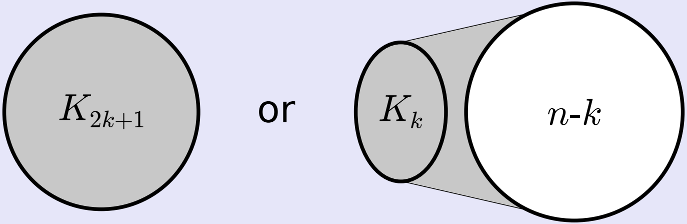

Observation

- When $k \approx \frac{n}{2} - 1$, process gives K_{2k+1}
- Last vertices consumed only at $n \log n$ edges
- At that point, everything densely connected

Results

Observation

- When $k = \frac{n}{9}$, process gives neither extremal graph
- Runs free for first $\frac{n}{9}$ edges, and until matching number hits k.
- Still linear number of edges at this time
- Constant probability of two isolated triangles; will never link

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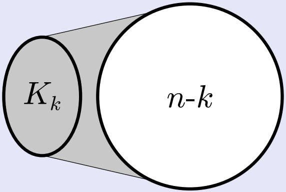
Theorem (Krivelevich-L.-Sudakov)

For $k = o(n)$, Matching Number Process a.a.s. gives



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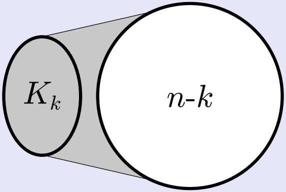

Remarks

- Global property
- Differential equations tracked local properties
- Proof introduces new methods using global statistics
- Completely solves problem up to $k = \Theta(n)$ threshold

Key Structure

Matching Structure

- Free run until matching number hits k, in $k + o(k)$ rounds
- Since $\mathbb{E}[\#$ edges that hit existing edge$] \leq \frac{k^2}{n} = o(k)$.
- Fix a matching of size k and build structure:

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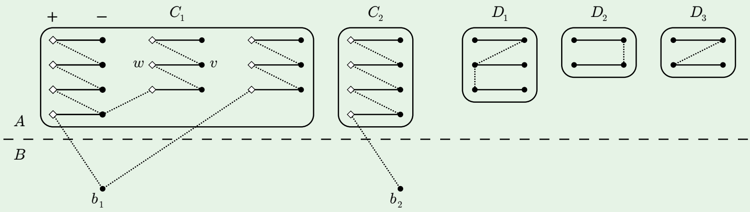

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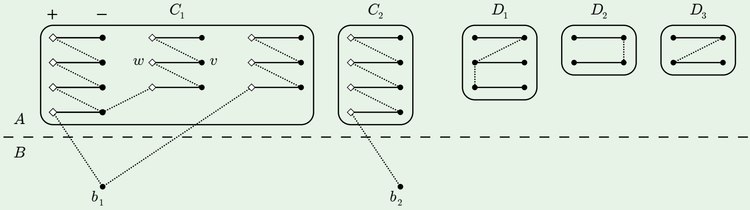

- If $C_i^- - D_j$ edge arrives, grow C_i by absorbing part of D_j

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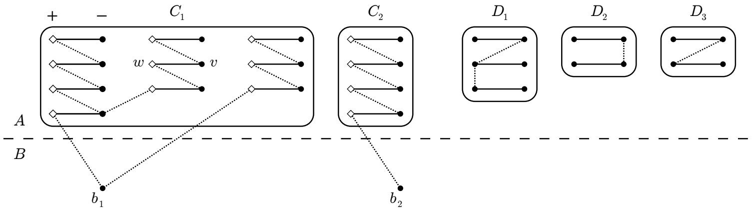

Observations

- Forever reject all $B - B$ edges

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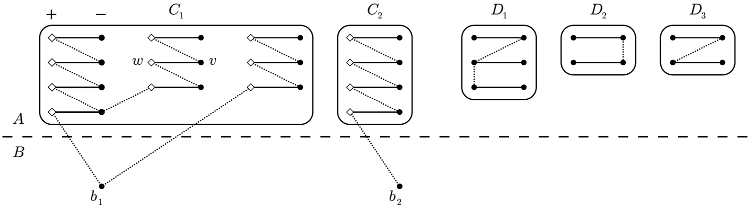

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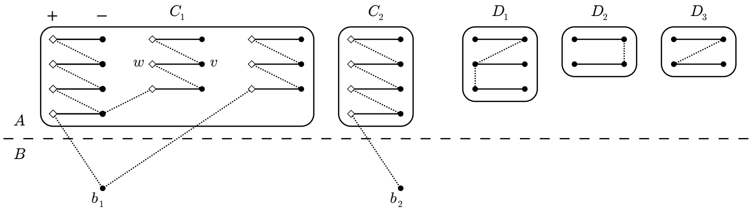

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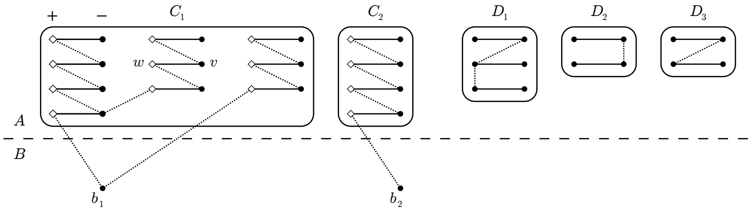

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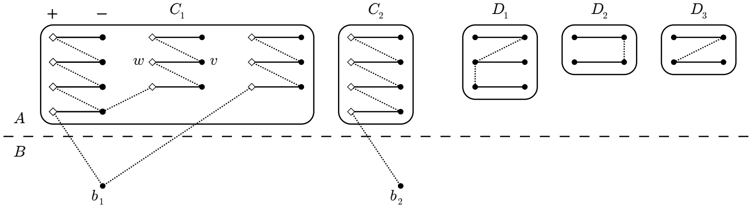
Key Structure Lemma

If $C_i^- \cup \{b_i\}$ independent, will accept any edge touching C_i^+ or D

Maintain Independence



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When v added to C^-, let S_v be all vertices already in C^-

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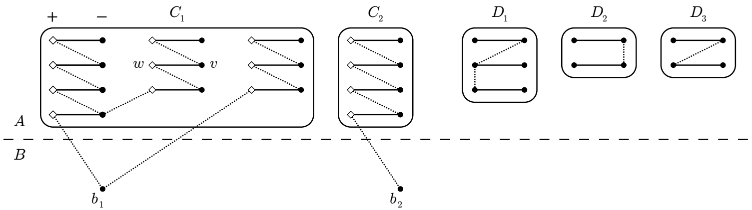

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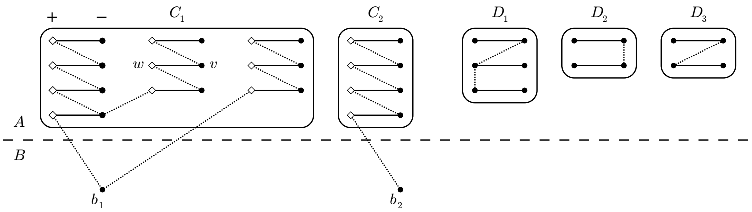

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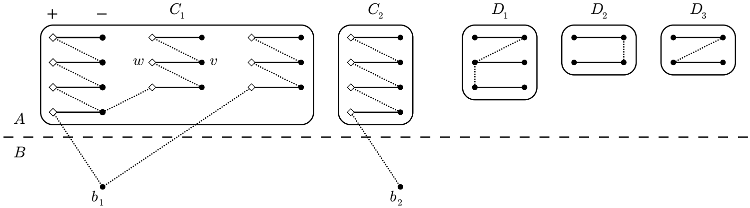

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- Cumulative bad probability $\leq \frac{\sum_v S_v}{n}$
- Susceptibility: at most $\frac{k}{n} = o(1)$