Monotone Paths

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Joint work with Mikhail Lavrov

Monotone sequences

Theorem (Erdős-Szekeres 1935)

Every permutation of $\{1, \ldots, n\}$ has a monotone subsequence of length about $\sqrt{n}$.

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Proof. Under each number, write lengths of longest increasing and decreasing subsequences ending there.

	1	**5**	**2**	**7**	**3**	**6**	**4**
inc.	1	2	2	3	3	4	4
dec.	1	1	2	1	2	2	3

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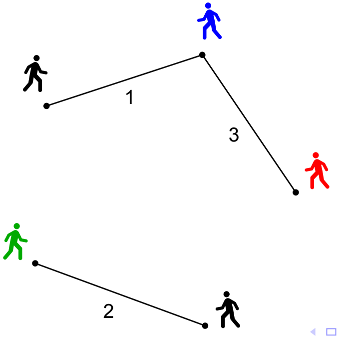

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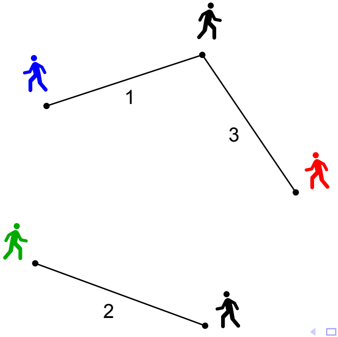

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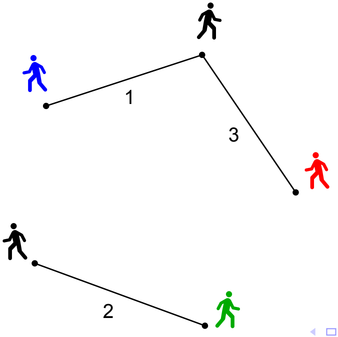

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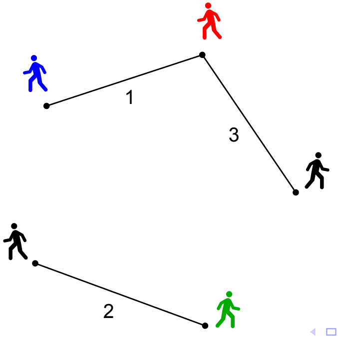

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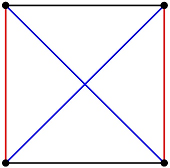

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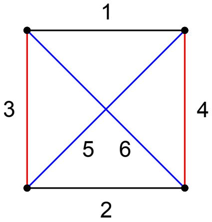

Assign a batch of consecutive labels to each matching.

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Self-avoiding walks are more complicated

- Easy poly-time algorithm to find longest increasing walk.
- In probability: self-avoiding random walk proven sub-ballistic only in 2012 by Duminil-Copin and Hammond.

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Theorem (Calderbank-Chung-Sturtevant 1984)

There is an edge-ordering of K_n in which the longest increasing path has length $(\frac{1}{2} - o(1))n$.

Random ordering

Model

Sample uniformly random ordering of $\binom{n}{2}$ edges.

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- Take that edge, then expose labels of edges to $n-2$ remaining vertices.
- Smallest increment is min of $n-2$ Unifs, so expectation $\frac{1}{n-1}$.

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- Take that edge, then expose labels of edges to $n - 2$ remaining vertices.
- Smallest increment is min of $n - 2$ Unifs, so expectation $\frac{1}{n-1}$.
- Sum $\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{cn} = 1$ when $\log \frac{1}{c} = 1$. □

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Trivial bound

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- For a given Hamiltonian path, it is increasing with probability $\frac{1}{(n-1)!}$.
- Number of Hamiltonian paths is $n!$.
- Expected number of increasing Hamiltonian paths is n . . .

First moment insufficient

$G_{n,p}$ has n Hamiltonian paths on expectation when $n!p^{n-1} \sim n$, i.e., when $p \sim \frac{e}{n}$.

In Erdős-Rényi

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A.a.s., random graph process gets Hamiltonian cycle at moment that all vertices have degree ≥ 2, which is at $p \sim \frac{\log n + \log\log n + \omega}{n}$.

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Theorem (Glebov-Krivelevich 2013)

At hitting time, number of Hamiltonian cycles jumps from 0 to $[(1+o(1))\frac{\log n}{e}]^n$ a.a.s.

Long increasing paths

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A random edge-ordering has an increasing Hamiltonian path with probability at least $\frac{1}{e}$.

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Conjecture (Lavrov, L.)

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Theorem (Chebyshev)

$$\mathbb{P}\left[|X - \mathbb{E}\left[X\right]| \geq t\right] \leq \frac{\mathrm{Var}\left[X\right]}{t^2}$$

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Theorem (Paley-Zygmund)

For nonnegative random variables X,

$$\mathbb{P}[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}$$

Calculation

Let $X = I_1 + \cdots + I_{n!}$, a sum with one indicator random variable per potential Hamiltonian increasing path.

$$\mathbb{E}\left[X^2\right] = \sum_{j,k} \mathbb{E}\left[I_j I_k\right]$$

Profiles

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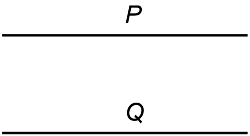

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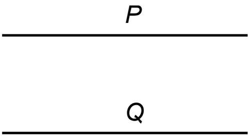

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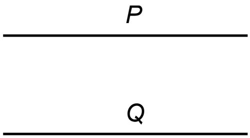

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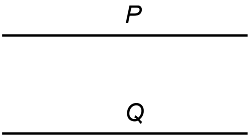

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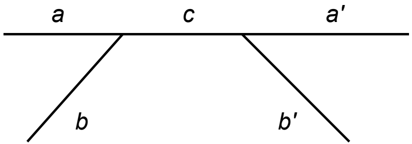
- Total contribution of profile: $n^2\frac{1}{e^2}$

Another easy profile



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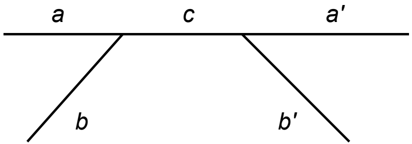

Probability

- Total number of edge labels: $a + b + c + a' + b'$.

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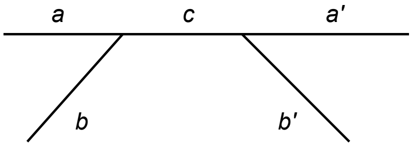

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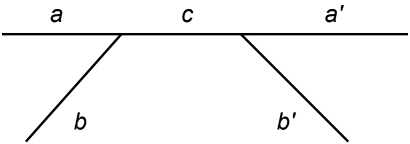

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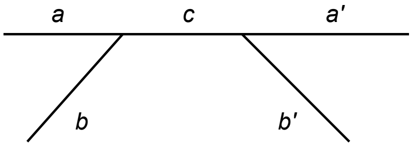

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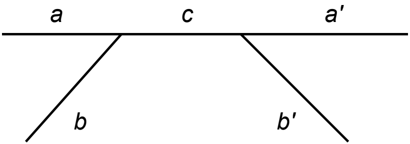

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$$\frac{\binom{a+b}{a}\binom{a'+b'}{a'}}{(a+b+c+a'+b')!}$$

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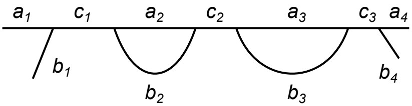

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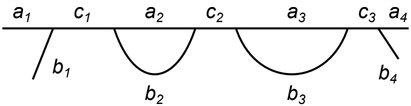

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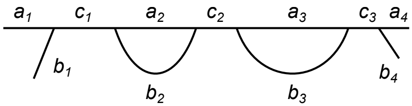

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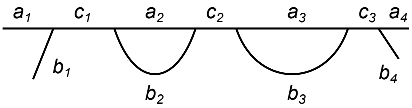

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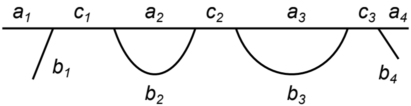

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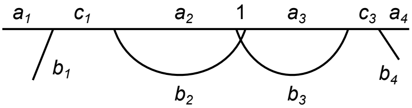
Number of embeddings

- Embed top path: $n!$
- Bottom path has $(c_1+1)+(c_2+1)+(c_3+1)$ vertices already fixed.
- Remaining vertices can be embedded in $(n-c_1-c_2-c_3-3)! \cdot e^{-2}$ ways.

General profile



Care required

When a common segment has length 1, i.e., some $c_i = 1$, the single common edge can also be traversed backwards.

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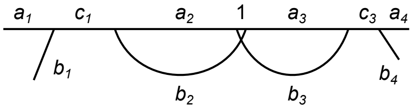

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- Number of embeddings is still $n!(n - c_1 - c_2 - c_3 - 3)! \cdot e^{-2}$.

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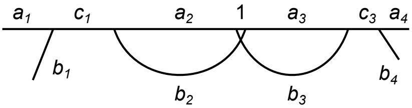

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- We pick up a factor of 2 for each $c_i = 1$.

Therefore, second moment of number of Hamilton increasing paths is $\mathbb{E}\left[X^2\right] =$

$$\sum_{\substack{a_1,a_2,\ldots\\ b_1,b_2,\ldots\\ c_1,c_2,\ldots}} n!\left[n-\sum(c_i+1)\right]!e^{-2}\cdot\frac{\prod\binom{a_i+b_i}{a_i}}{\left[\sum a_i+\sum b_i+\sum c_i\right]!}\cdot 2^{\#\{i:c_i=1\}}$$

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which, after some work, turns out to be $(1+o(1))en^2$.

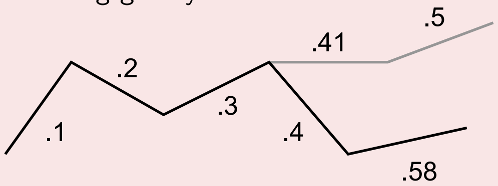
Cost of greed

Greedy algorithm

Always pick edge with smallest increment to a new vertex.

Potential gain

Consider the following greedy outcome:



Greedy algorithm, with temptation

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- Repeat until exploration tree has k edges.

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- Extend path to earliest subtree.
- Replace exploration tree by that subtree, and repeat.

k-GREEDY ALGORITHM

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- When extending path, do not immediately pick smallest increment.
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- Repeat until exploration tree has k edges.
- Extend path to largest subtree.
- Replace exploration tree by that subtree, and repeat.

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Time to grow path from $\ell \to \ell + 1$

- Suppose exploration tree has t vertices, and path has length ℓ.

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Typical time to grow path from $0 \to \ell$

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Golomb-Dickman constant

If T_k is largest table after k people, then

$$\mathbb{E}\left[\frac{T_k}{k}\right] \to 0.6243$$

Calculation for k-greedy

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- Typical length is when

$$\log \frac{n}{n-\ell} = \frac{1}{0.5219} \quad \Rightarrow \quad \ell = (1 - e^{-1/0.5219})n.$$

Concluding remarks

Theorem (Lavrov, L.)

- A random edge-ordering has an increasing Hamiltonian path with probability at least $\frac{1}{e}$.
- With backtracking, k-greedy algorithm finds an increasing path of length $0.85n$ a.a.s. in a random edge-ordering.
- Let X be the number of Hamiltonian increasing paths. Then $\mathbb{E}\left[X^2\right] \sim en^2$.

Conjecture (Lavrov, L.)

A random edge-ordering has an increasing Hamiltonian path a.a.s.