Judicious bisections

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Joint work with Choongbum Lee and Benny Sudakov

Max Cut

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Given a graph, find a bipartition which maximizes the number of crossing edges.

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Proof:

- Independently choose a random side for each vertex.
- Each particular edge then crosses with probability $\frac{1}{2}$.
- Expected number of crossing edges is exactly half. $\square$

Extensions

Let n and m be the numbers of vertices and edges, respectively.

Theorem (Edwards '73)

Every graph has a cut of size at least $\frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8}$.

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Theorem (Erdős-Gyárfás-Kohayakawa '97)

For graphs with no isolated vertices, Max Cut $\geq \frac{m}{2} + \frac{n}{6}$.

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Theorem (Bollobás-Scott '99)

All graphs admit bipartitions that achieve the Edwards bound, while also inducing at most $\frac{m}{4} + \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16}$ edges per side.

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A bipartition into two parts of equal size is called a *bisection*.

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Do judicious bisections always exist, inducing $\leq \frac{m}{4}$ edges per side?

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Question (Bollobás-Scott)

What is the best result for graphs of minimum degree at least δ?

Main results

Theorem (Lee-L.-Sudakov)

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For even integers δ, graphs with all degrees $\geq \delta$ admit bisections that induce fewer than $\left(\frac{\delta+2}{4(\delta+1)} + o(1)\right)m$ edges per side.

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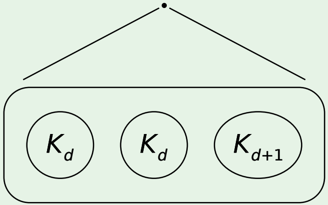
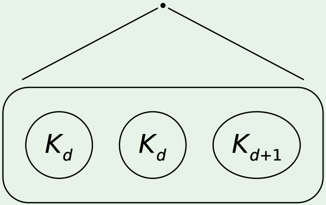
Remark. For odd δ, the bound for $\delta-1$ applies.

Construction

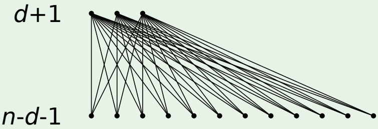


Tightness

Construction



Construction 2



Second moment method

Randomized bisection algorithm

- Arbitrarily pair up vertices.
- Independently split each pair randomly.

Analysis.

- Let X and Y be the numbers of edges induced by each side.
- $\mathbb{E}[X]$ is at most $\frac{m}{4}$.

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- Chebyshev: X exceeds $\mathbb{E}[X]$ by $2\sqrt{\mathrm{Var}(X)}$ with probability at most $\frac{1}{4}$.
- There is an outcome with **both** $X, Y \leq \frac{m}{4} + 2\sqrt{\sum d(v)^2}$.

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- $\sum d(v)^2 \leq 2m\Delta \leq 2mn$.

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Theorem (Lee-L.-Sudakov)

For graphs with no isolated vertices and maximum degree $\leq \frac{n}{3} + 1$,

$$\text{Max Bisection} \geq \frac{m}{2} + \frac{n}{6}.$$

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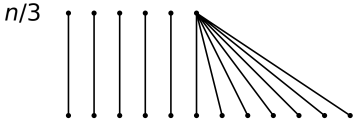
Remark. Matches Max Cut bound; max degree condition is tight.



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- If d is the number of new edges contributed by adding u and v, then one orientation produces at least $\frac{d}{2}$ new crossing edges. □

Observations.

- If the new pair $\{u, v\}$ is actually an edge, then guarantee improves by $+\frac{1}{2}$.

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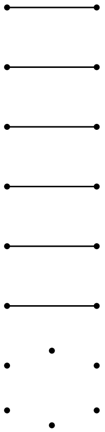
- If the new pair $\{u, v\}$ is actually an edge, then guarantee improves by $+\frac{1}{2}$.
- Also gain $+\frac{1}{2}$ if d is odd.

Proof of $\frac{m}{2} + \frac{n}{6}$ bound.



- Take maximal matching; if more than $\frac{n}{3}$ edges, done.

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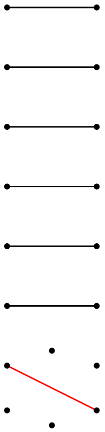

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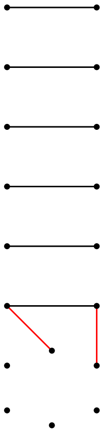

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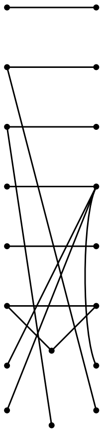

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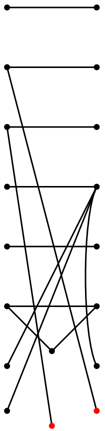

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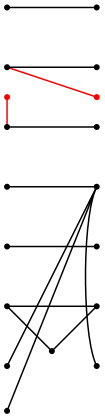

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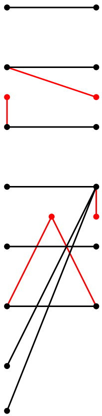

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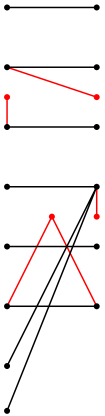

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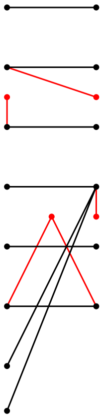

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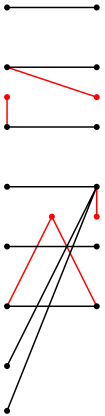

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- All unpaired vertices form triangles to matching, or all unpaired vertices have a common neighbor.
- In both cases, there are at most $\frac{n}{3}$ unpaired vertices, hence at least $\frac{n}{3}$ pairs that are edges or odd-backs.
- Greedy algorithm gives bisection of size at least $\frac{m}{2} + \frac{n}{6}$. □

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