

# Pidgeonhole Principal

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## 1 Warm-up

1. How do you spell this title?

**Solution:** If you can't spell this, you shouldn't be a discreet mathematician.

2. Given  $n$  integers, prove that some nonempty subset of them has sum divisible by  $n$ .

**Solution:** Let the integers be  $x_1, \dots, x_n$ , and consider the partial sums. Assuming that none of the partial sums are divisible by  $n$ , there must be a repeated partial sum; their difference is a sum divisible by  $n$ .

3. The USAMO awards banquet at the U.S. Department of State has assigned seating.<sup>1</sup> However, after sitting down, the table #1 group finds that by mistake, they have all sat down at the wrong seats (although their correct seat is somewhere else at that table). Show that it is possible for them to simultaneously rotate around the table by some distance, such that at least two of them end up in the correct place.

**Solution:** Let  $n$  be the number of people. For each person, note down the clockwise distance to their correct seat. Since all are wrong, these are integers from 1 to  $n - 1$ , so some integer appears at least twice.

## 2 Solve any 15 of these problems

1. (Erdős-Szekeres upper bound.) The *Ramsey Number*  $R(s, t)$  is the minimum integer  $n$  for which **every** red-blue coloring of the edges of  $K_n$  contains a completely red  $K_s$  or a completely blue  $K_t$ . Prove that

$$R(s, t) \leq \binom{s+t-2}{s-1}$$

**Solution:** Observe that  $R(s, t) \leq R(s-1, t) + R(s, t-1)$ , because if we have that many vertices, then if we select one vertex, then it cannot simultaneously have  $< R(s-1, t)$  red neighbors and  $< R(s, t-1)$  blue neighbors, so we can inductively build either a red  $K_s$  or a blue  $K_t$ . But

$$\binom{(s-1)+t-2}{(s-2)} + \binom{s+(t-1)-2}{s-1} = \binom{s+t-2}{s-1},$$

because in Pascal's Triangle the sum of two adjacent guys in a row equals the guy directly below them in the next row.

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<sup>1</sup>This is true. But the rest of the problem setting is purely fictional.

2. (Paul Erdős.) Let  $A$  be a set of  $n + 1$  integers from  $\{1, \dots, 2n\}$ . Prove that some element of  $A$  divides another.

**Solution:** Partition  $[2n]$  into  $n$  sets, such that each set has the property that for any two of its elements, one divides the other. This can be done as follows. Take  $\{1, 2, 4, 8, \dots\}$ ,  $\{3, 6, 12, \dots\}$ ,  $\dots$ , where we start with an odd number, and then keep multiplying by 2. We never need to start with an even number, since they are automatically covered by multiplying an earlier odd number by 2. So they partition  $[2n]$ , as desired.

3. (Trivial approximation.) For  $x \in \mathbb{R}$  and  $n \in \mathbb{Z}^+$ , there is a rational number  $\frac{p}{q}$ , with  $1 \leq q \leq n$ , such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{n}.$$

**Solution:** Pick  $q = n$  and let  $p$  range over the integers.

4. (Dirichlet approximation.) Same as above, except with

$$\left| x - \frac{p}{q} \right| < \frac{1}{nq}.$$

**Solution:** Consider the numbers  $\{ax\}$ , where this is the fractional part, and  $a$  runs from 0 to  $n$ . Consider the  $n$  buckets  $[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), \dots$ . By pigeonhole, some bucket contains both  $\{ax\}$  and  $\{a'x\}$ , say with  $\{ax\} < \{a'x\}$ . This means there is some integer  $p$  such that  $0 < a'x - ax - p < \frac{1}{n}$ . Let  $q = a' - a$  and divide through by it, and we are done.

5. (Kronecker's Theorem.) Let  $x$  be an irrational number, and let  $x_n = \{nx\}$  be the fractional part of  $nx$ . Show that the sequence  $x_1, x_2, \dots$  is *dense* in the interval  $[0, 1)$ . This means that for every real number  $r \in [0, 1)$ , and every  $\epsilon > 0$ , there is some  $n$  such that  $x_n$  is within  $\epsilon$  of  $r$ .

**Solution:** It suffices to show: Given any integer  $q$  and any integer  $p$  between 0 and  $q$ , eventually  $x_n$  falls in the interval between  $\frac{p}{q}$  and  $\frac{p+1}{q}$ . Consider the  $q$  buckets  $[0, \frac{1}{q}), [\frac{1}{q}, \frac{2}{q}), \dots$ . Eventually some  $x_n$  and  $x_{n'}$  end up in the same bucket, meaning that the fractional part  $\{(n' - n)x\}$  has absolute value less than  $\frac{1}{q}$ . Then consider  $x_{k(n' - n)}$  for positive integers  $k$ , and this will walk through the buckets (never skipping, but possibly backwards) until it ends up in the interval  $[\frac{p}{q}, \frac{p+1}{q})$ .

This implies density, because given the  $r$  and  $\epsilon$ , we can choose sufficiently large  $q$  such that the entire interval  $[\frac{p}{q}, \frac{p+1}{q})$  is contained within  $r \pm \epsilon$ .

6. (Erdős-Szekeres.) Prove that every sequence of  $n^2$  distinct numbers contains a subsequence of length  $n$  which is monotone (i.e. either always increasing or always decreasing).

**Solution:** For each of the  $n^2$  indices in the sequence, associate the ordered pair  $(x, y)$  where  $x$  is the length of the longest increasing subsequence ending at  $x$ , and  $y$  is the length of the longest decreasing one. All ordered pairs must obviously be distinct. But if they only take values with  $x, y \in \{1, \dots, n-1\}$ , then there are not enough for the total  $n^2$  ordered pairs. Thus  $n$  appears somewhere, and we are done.

7. Given any 7 points inside a hexagon with side length 1, show that some two points are separated by a distance of at most 1.

**Solution:** Cut the hexagon with the 3 main diagonals, to get 6 triangles with side length 1. Two of the points lie in the same triangle.

8. (USAMO 1976.) Every square in a  $4 \times 7$  array is colored either white or black. Show that there always is a monochromatic "constellation" consisting of the 4 corners of an axis-parallel rectangle.

**Solution:** Key observation:  $7 = \binom{4}{2} + 1$ . First, suppose that some column contains 3 or more of the same color, say black. WLOG, they are in the first 3 rows. Then, for the other 6 columns, there

cannot ever be 2 black in the first 3 rows, i.e., there must be at least 2 white in the first 3 rows. The number of ways to do this is all white (1), 2 white (3), so by the 5th column, there is repetition, and this gives the 4 corners.

Thus, in every column there are exactly 2 of each color. But then after  $\binom{4}{2}$  columns, there is a repeat, so we get the 4 corners again.

9. Let  $S$  be a set of 10 positive integers, whose total sum is less than 250. Prove that there exist two disjoint nonempty equal-size subsets  $A, B \subset S$  such that the sum of the elements of  $A$  equals the sum of the elements of  $B$ .

**Solution:** Consider all 5-subsets of  $S$ . There are  $\binom{10}{5} = 252$  of them, so some pair has the same sum. To make them disjoint, remove all common elements. The resulting sets will have the same size, as desired.

10. (MOP 2004.) A set  $S$  of numbers is called a *Sidon set* if it has the property that for every distinct  $a, b, c, d \in S$ , the sums  $a + b$  and  $c + d$  are distinct. (There are no repeated pairwise sums between elements of  $S$ .) A natural question is to ask how large a Sidon set can be, if, say, the numbers must be integers in  $\{1, \dots, 100\}$ . Prove that there is no such Sidon set of size 16.

**Solution:** The key observation is that if we find  $a - c = b - d$ , for distinct  $a, b, c, d$ , then this is equivalent. There are  $\binom{16}{2} = 120$  pairs of elements from  $S$ , so if we take the larger minus the smaller, we get 120 positive differences. But there are only 99 possible positive differences, so we repeat. If the repetition uses distinct  $a, b, c, d$ , then we are done. The only remaining issue is if the repetition is of the form  $c - b = b - a$ . This corresponds to a 3-term arithmetic progression:  $(a, b, c)$ .

Let us bound the number of 3-AP's in  $S$ . If an element  $x \in S$  is the central term in two different 3-AP's, say  $(a, x, b)$  and  $(c, x, d)$ , then we already have  $a + d = b + c$ , done. So, each of the 16 elements of  $S$  can only contribute at most one 3-AP.

To carefully account for everything, now create 99 holes for the possible positive differences. Iteratively run through the  $\binom{16}{2}$  pairs, and put them into the holes. There will be at least 21 times that we put the new item into a hole already containing some pigeon(s). Suppose this is the case. If there are already 2 pigeons in the hole, then they must correspond to a 3-AP, and the new pigeon can combine with one of the other two to make the desired  $a - c = b - d$  with distinct  $a, b, c, d$ . If there is only 1 pigeon, and it uses distinct elements, then we win as above too. The only other possibility is that there is 1 pigeon, and it forms a 3-AP with the incoming pigeon. This situation can only happen 16 times, so there will be some instance of the previous two types.

11. (Putnam 1993.) Let  $x_1, \dots, x_{19}$  be positive integers less than or equal to 93. Let  $y_1, \dots, y_{93}$  be positive integers less than or equal to 19. Prove that there exists a (nonempty) sum of some  $x_i$ 's equal to a sum of some  $y_i$ 's.

**Solution:** Suppose false. WLOG,  $\sum y_i > \sum x_i$ . For each partial sum of  $x$ 's, consider the shortest partial sum of  $y$ 's that has greater sum. The difference between the  $y$ -partial sum and the  $x$ -partial sum is then in  $\{1, \dots, 18\}$ . (18 because we assume it cannot be that the partial sums were equal.) But there are 19 partial sums of  $x$ 's, so there is a repeat. Then the corresponding two intermediate sums of  $x$ 's and  $y$ 's are equal.

12. A  $100 \times 100$  array is filled with numbers from  $\{1, \dots, 100\}$ , such that each number appears exactly 100 times. Prove that there is some row or column which contains at least 10 different numbers.

**Solution:** Linearity of expectation with indicator variables. Translated to pigeonhole: 200 holes are the rows and columns, which we call *lines*. A Pigeon is a pair (number, line) such that the number appears in the line. To count the number of pigeons, note that for each fixed number  $i$ , the minimum number of lines that it can appear in is 20 by AM-GM. So, there are at least 2000 pigeons, which means that some hole has at least 10 of them.

13. The Fibonacci numbers are defined by  $F_1 = F_2 = 1$  and  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 3$ . If  $p$  is a prime number, prove that at least one of the first  $p + 1$  Fibonacci numbers must be divisible by  $p$ .
14. (Putnam 2001.) Let  $B$  be a set of more than  $2^{n+1}/n$  distinct points in  $n$ -dimensional space with coordinates of the form  $(\pm 1, \dots, \pm 1)$ , where  $n \geq 3$ . Show that there are 3 distinct points in  $B$  which are the vertices of an equilateral triangle.