Extremal Graph Theory

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Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.

(The opening sentence in *Extremal Graph Theory*, by Béla Bollobás.)

1 Warm-up

1. What is the smallest possible number of edges in a *connected* n-vertex graph?

   Solution: By breadth-first exploration, we always can find a spanning tree in every connected graph. This process starts with 1 vertex and 0 edges, but adds 1 vertex and edge at each iteration. So we end up with $n-1$ edges in our spanning tree.

2. (Reid Barton, 2005.) There are n pieces of candy in a pile. One is allowed to separate a pile into two piles, and add the product of the sizes of the two new piles to a running total. The process terminates when each piece of candy is in its own pile. Show that the final sum is independent of the order of the operations performed.

   Solution: This will be the number of edges in the complete graph K_n, i.e., $\binom{n}{2}$. To see this, run the process in reverse. We start with n independent vertices (each in their own cluster). Note that each cluster is a complete graph. Each round, we pick two disjoint clusters and merge them, adding the edges of the complete bipartite graph between the clusters. The number of added edges is precisely the product of the cluster sizes. But we preserve the fact that every cluster is a complete graph, so at the end we have a single K_n.

2 Extremal graph theory

This very interesting field happens to be the subject of my own research, as well as one of the most common sources of advanced graph theory problems in Olympiads. The most famous theorems concern what substructures can be forced to exist in a graph simply by controlling the total number of edges. The classical starting point is Turán's theorem, which proves the extremality of the following graph: let $T_r(n)$ be the complete r-partite graph with its n vertices distributed among its r parts as evenly as possible (because rounding errors may occur).

Theorem. *(Turán) For $r \geq 3$, the Turán graph $T_{r-1}(n)$ is the unique n-vertex graph with the maximum number of edges subject to having no K_r subgraphs.*

An excellent proof of Turán's theorem can be found on page 167 of the book *Graph Theory*, by Reinhard Diestel. This is a well-written book which has an electronic edition **freely available** on the author's website!

2.1 More subgraph results

1. (Quantitative form of Turán for triangles.) Turán's Theorem implies that every graph with average degree greater than $n/2$ contains a triangle. Show that every graph with average degree at least $\left(\frac{1}{2}+c\right)n$ contains at least $c\binom{n}{3}$ triangles.

   Solution: We actually show that it contains at least $\frac{c}{3}n^2 d$ triangles. Note that when d goes up to n, then c goes to $\frac{1}{2}$, which would give roughly $\frac{1}{6}n^3$, the maximum possible.

   The number of triangles containing a given edge uv is at least $d_u + d_v - n$, so summing over all edges gives that the number of triangles is at least

$$
\begin{aligned}
\frac{1}{3}\sum_e d_u + d_v - n &= \frac{1}{3}\left(\sum_v d_v^2 - nE\right) \\
&\geq \frac{1}{3}\left(nd^2 - n\frac{nd}{2}\right) \\
&= \frac{1}{3}nd\left(d - \frac{n}{2}\right) \\
&\geq \frac{1}{3}ndcn
\end{aligned}
$$

   The result follows since $d \geq \frac{n}{2}$.

2. Every graph G with average degree d contains a subgraph H such that all vertices of H have degree at least $d/2$ (with respect to H).

   Solution: Condition on G is that the number of edges is at least $nd/2$. If there is a vertex with degree $< d/2$, then delete it, and it costs 1 vertex and $< d/2$ edges, so the condition is preserved. But it can't go on forever, because once there is 1 vertex left, average degree is 0.

3. (approximation to Erdős-Sós conjecture) Let T be a tree with t edges. Then every graph with average degree at least $2t$ contains T as a subgraph.

   Solution: Graph has subgraph with minimum degree at least t. Then embed greedily. Suppose we already put down v vertices. ($v < t+1$ or else we are done.) Pick a current node to which to adjoin a new leaf. Degree is at least t, and $v-1$ vertices are already down (so blocked for embedding), so $t - v + 1 > 0$ choices remain. Pick one of them for the new leaf, and continue.

4. We say that a graph G is *t-degenerate* if every subgraph has a vertex of degree $\leq t$. Show that G can be properly colored with $\leq t+1$ colors.

   Solution: Iteratively peel off vertex of degree $\leq t$, and put these into an ordering. That is, v_1 is the first vertex pulled off, then v_2, etc. Now greedily color from v_n to v_1.

5. Suppose that a certain country has the property that it is not possible to color a map of its states using only 3 colors, such that no adjacent states receive the same color. Prove that there exist 4 states, A, B, C, and D, such that $A \sim B$, $B \sim C$, and $C \sim D$, where the "$\sim$" symbol means that two states are adjacent.

   Solution: This is a special case of the following result: Let T be a tree with t edges. Then every graph with chromatic number greater than t contains a copy of T. To see this, note that the contrapositive of previous is that $\chi \geq t+2$ implies there is a subgraph with minimum degree $\geq t+1$. Actually we need it with $\chi \geq t+1$ implying subgraph with minimum degree $\geq t$. Then embed tree greedily as in the previous solution.

2.2 Ramsey theory

Complete disorder is impossible.

— T. S. Motzkin, on the theme of Ramsey Theory.

The *Ramsey Number* $R(s,t)$ is the minimum integer n for which **every** red-blue coloring of the edges of K_n contains a completely red K_s or a completely blue K_t. Ramsey's Theorem states that $R(s,t)$ is always finite, and we will prove this in the first exercise below. The interesting question in this field is to find upper and lower bounds for these numbers, as well as for quantities defined in a similar spirit.

Now try these.

1. Prove by induction that $R(s,t) \le \binom{s+t-2}{s-1}$. Note that in particular, $R(3,3) \le 6$.

   Solution: Observe that $R(s,t) \le R(s-1,t) + R(s,t-1)$, because if we have that many vertices, then if we select one vertex, then it cannot simultaneously have $< R(s-1,t)$ red neighbors and $< R(s,t-1)$ blue neighbors, so we can inductively build either a red K_s or a blue K_t. But

   $$\binom{(s-1)+t-2}{(s-2)} + \binom{s+(t-1)-2}{s-1} = \binom{s+t-2}{s-1},$$

   because in Pascal's Triangle the sum of two adjacent guys in a row equals the guy directly below them in the next row.

2. Show that $R(t,t) \le 2^{2t}$. Then show that $R(t,t) > 2^{t/2}$ for $t \ge 3$, i.e., there is a red-blue coloring of the edges of the complete graph on $2^{t/2}$, such that there are no monochromatic K_t.

   Solution: The first bound follows immediately from the Erdős-Szekeres bound. The second is an application of the probabilistic method. Let $n = 2^{t/2}$, and consider a random coloring of the edges of K_n, where each edge independently receives its color with equal probabilities. For each set S of t vertices, define the event E_S to be when all $\binom{t}{2}$ edges in S are the same color. It suffices to show that $\mathbb{P}\left[\text{some } E_S \text{ occurs}\right] < 1$. But by the union bound, the LHS is

   $$\begin{aligned} \binom{n}{t} \cdot \left(2 \cdot 2^{-\binom{t}{2}}\right) &\le \frac{n^t}{t!} \cdot 2 \cdot 2^{-\frac{t^2}{2}+\frac{t}{2}} \\ &= \frac{\left(2^{t/2}\right)^t}{t!} \cdot 2 \cdot 2^{-\frac{t^2}{2}+\frac{t}{2}} \\ &= 2 \cdot \frac{2^{t/2}}{t!}. \end{aligned}$$

   This final quantity is less than 1 for all $t \ge 3$.

3. (IMO 1964/4) Seventeen people correspond by mail with one another—each one with all the rest. In their letters only 3 different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least 3 people who write to each other about the same topic.

   Solution: This is asking us to prove that the 3-color Ramsey Number $R(3,3,3)$ is ≤ 17. By the same observation as in the previous problem, $R(a,b,c) \le R(a-1,b,c) + R(a,b-1,c) + R(a,b,c-1) - 1$. Then using symmetry, $R(3,3,3) \le 3R(3,3,2) - 1$. It suffices to show that $R(3,3,2) \le 6$. But this is immediate, because if we have 6 vertices, if we even use the 3rd color on a single edge, we already get a K_2. So we cannot use the 3rd color. But then from above, we know $R(3,3) \le 6$, so we are done.

3 Problems

1. (Japan 1998/2) A country has 1998 airports connected by some direct flights. For any three airports, some two are not connected by a direct flight. What is the maximum number of direct flights that can be offered?

Solution: By Turán, the largest triangle-free subgraph of K_{1998} is bipartite with sides of size 999, so 999^2 is the answer.

2. (South Africa 1997/5) Six points are joined pairwise by red or blue segments. Must there exist a closed path consisting of four of the segments, all of the same color?

   Solution: Yes. Proof: assume not. Let the vertices be a, b, c, d, e, f. Use Ramsey to get a monochromatic triangle first, and suppose it is a, b, c. WLOG it is blue. Now d cannot have 2 blue edges into a, b, c, or else we get blue C_4. Same for e and f. Also, if, say, a, b both have red edges to each of say, d, e, then we get C_4. Therefore, the only possible configuration is to have a blue matching between $\{a, b, c\}$ and $\{d, e, f\}$, and all other edges between those sets are red. WLOG the blue matching is ad, be, cf. But then edges de and ef are both forced red, or else we have blue C_4, say $abed$. And then there is a red C_4: $bfed$.

3. (Japan 1997/3) Let G be a graph with 9 vertices. Suppose given any five points of G, there exist at least 2 edges with both endpoints among the five points. What is the minimum possible number of edges in G?

   Solution: The optimal configuration is a union of 3 disjoint triangles. Let a_n be the minimum number of edges of a graph on n vertices satisfying the condition. We will need to show that $a_{n+1} \geq \frac{n+1}{n-1} a_n$. Then, since $a_5 = 2$, we will have $a_6 \geq 3$, $a_7 \geq 5$, $a_8 \geq 7$, and $a_9 \geq 9$.

   Now prove the inequality by induction. Consider any $(n+1)$-vertex graph satisfying the property, with E edges. Deleting any vertex gives an n-vertex graph with the property, with $E - d_v$ edges left, so by induction $E - d_v \geq a_n$. Sum over all $n + 1$ vertices v, and we get $(n + 1)E - 2E \geq (n + 1)a_n$, so $E \geq \frac{n+1}{n-1} a_n$ as desired.

4 Really harder problems

Determine the exact value of the Ramsey number $R(5,5)$. Hint: it is known to be one of $\{43, 44, 45, 46, 47, 48, 49\}$. You may use as many supercomputers as you want.

Believe it or not, this is unknown. For a greater challenge, determine $R(6, 6)$. If you succeed, Paul Erdős would have been proud.

> *Imagine an alien force, vastly more powerful than us landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they asked for $R(6,6)$, we should attempt to destroy the aliens.*

-Paul Erdős