Graph Theory

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At first, graph theory may seem to be an *ad hoc* subject, and in fact the elementary results have proofs of that nature. The methods recur, however, and the way to learn them is to work on problems. Later, when you see an Olympiad graph theory problem, hopefully you will be sufficiently familiar with graph-theoretic arguments that you can rely on your own ingenuity to invent an approach.

1 Warm-up

1. Graph the function $f(x) = \frac{x-2}{x+3}$.

2. There are 21 friends at a party. Each person counts how many of the other people are he/she has met before. Is it possible for the sum of these counts to be an odd number?

2 Terminology

Definition. *A* ***graph*** $G = (V, E)$ *is a collection* V *of* ***vertices*** *and* $E \subset V \times V$ *of* ***edges****.*

Remark. Informally, we think of the edges as linking the pairs of vertices that they correspond to, and typically represents graphs by drawings in which we connect the endpoints by a curve.

Another remark. If every edge links a unique pair of distinct vertices, then we say that the graph is *simple*. Most of our work will be with simple graphs, so we usually will not point this out.

Glossary of terms.

- Two vertices are *adjacent* if there is an edge that has them as endpoints.
- A graph is *bipartite* if the vertex set can be partitioned into two sets $V_1 \cup V_2$ such that edges only run between V_1 and V_2.
- The *chromatic number* of a graph is the minimum number of colors needed to color the vertices without giving the same color to any two adjacent vertices.
- A *clique* on n vertices, denoted K_n, is the n-vertex graph with all $\binom{n}{2}$ possible edges.
- A *complete* graph on n vertices, denoted K_n, is the n-vertex graph with all $\binom{n}{2}$ possible edges.
- A graph is *connected* if there is a path between every pair of distinct vertices.
- A *cycle* is a path for which the first and last vertices are actually adjacent.
- The *degree* $d(v)$ of a vertex v is the number of edges that are incident to v.
- An *Eulerian circuit* is a walk that traverses every edge exactly once, and returns to its starting point.

- A *forest* is a not-necessarily-connected graph with no cycles. See also the definition of *tree* below.
- If U is a subset of the vertices, then the *induced subgraph* $G[U]$ is the graph obtained by deleting all vertices outside U, keeping only edges with both endpoints in U.
- A *Hamiltonian path* is a path that includes every vertex. A *Hamiltonian cycle* is a cycle that includes every vertex.
- We say that an edge e is *incident* to a vertex v if v is an endpoint of e.
- A *path* is a sequence of distinct, pairwise-adjacent vertices. See also the definition of *walk* below.
- A graph is *planar* if it is possible to draw it in the plane without any crossing edges.
- A *tree* is a connected graph with no cycles. See also the definition of *forest* above.
- A *walk* is a sequence of not-necessarily-distinct, pairwise-adjacent vertices. See also the definition of *path* above.

3 Tools

3.1 Bare-hands

Each of these well-known results can be proved without any fancy theorems. **Prove them.**

1. The sum of all of the degrees is equal to twice the number of edges. Deduce that the number of odd-degree vertices is always an even number.

   Solution: By counting in two ways, we see that the sum of all degrees equals twice the number of edges.

2. Every tree contains a vertex of degree exactly 1, which is called a *leaf.*

   Solution: If all vertices had degree ≥ 2, then we could take a walk around the graph (with the rule that we are not allowed to go back over the edge that we took in the previous step), and then we will eventually intersect our path again. This forms a cycle.

3. Every tree can be constructed by starting with a single vertex, and adding one new leaf at a time.

   Solution: Since every tree has a leaf, we can "destruct" any tree by pulling off a leaf. Reversing these steps gives a construction.

4. Every connected graph contains a *spanning tree.* This is a subgraph which is a tree, that includes all of the original vertices.

   Solution: Keep deleting one edge from a remaining cycle until all cycles are gone. This cannot hurt connectedness, so we end up with a tree. Alternatively, consider the breadth-first-search or depth-first-search.

5. A connected graph is a tree if and only if it has exactly $V - 1$ edges.

   Solution: Forward implication follows from the construction of a tree by adding one leaf at a time. Reverse implication follows by considering a spanning tree inside the connected graph, and noting that it already consumes all of the edges.

6. The vertices can be partitioned into sets $V_1 \cup \ldots \cup V_r$, where each induced subgraph $G[V_i]$ is a connected graph, and there are no edges between any pair of distinct $\{V_i, V_j\}$. These are called the *connected components* of G.

   Solution: Trivial observation.

7. A graph is bipartite if and only if it has no odd cycles.

   Solution: Separate into connected components. For each, choose a special vertex, and color based on parity of length of shortest path from that special vertex.

8. Every connected graph with all degrees even has an Eulerian circuit.

   Solution: Start walking from a vertex v_1 without repeating any edges, and observe that by the parity condition, the walk can only get stuck at v_1, so we get one cycle. If we still have more edges left to hit, connectivity implies that some vertex v_2 on our current walk is adjacent to an unused edge, so start the process again from v_2. Splice the two walks together at v_2, and repeat until done.

9. The chromatic number of a graph is always $\leq$ its maximum degree plus one.

   Solution: Consider the greedy algorithm for coloring vertices.

Try these problems to get a feel for graph theoretic arguments.

1. 2-colorable graphs contain no odd cycles.

   Solution: Observe that bipartite graphs and 2-colorable graphs are the same thing.

2. What do graphs with all degrees ≤ 2 look like?

   Solution: Disjoint union of isolated vertices, cycles, and paths.

3. Let G be a graph. It is possible to partition the vertices into two groups such that for each vertex, at least half of its neighbors ended up in the other group.

   Solution: Take a max-cut: the bipartition which maximizes the number of crossing edges.

4. (Diestel 1.16) Let G be a tree, and let Δ be its maximum degree. Show that G has at least Δ leaves.

   Solution: Take the vertex v with degree Δ, and choose a leaf from each subtree rooted at each neighbor of v.

5. Let δ be the minimum degree of G, and suppose that $\delta \geq 2$. Then G contains a cycle of size $\geq \delta + 1$. In particular, it contains a path with $\geq \delta$ edges.

   Solution: Take a longest path. Let v be its last endpoint. By maximality, every one of v's $\geq \delta$ neighbors lie on the path. So path has length $\geq \delta + 1$.

6. Prove that every graph with maximum degree ≤ 3 has a (2,2)-relaxed coloring. That means there is a way to color the vertices red or blue such that:

   - in the graph induced by the red vertices, all connected components have size ≤ 2, and
   - similarly for the graph induced by the blue vertices.

   Solution: Take a max-cut.

7. (Hard, Diestel 1.7) If G is also known to be connected, G actually contains a path with $\geq \min\{2\delta, V-1\}$ edges.

Now here are some actual Olympiad problems to try.

1. (Tournament of the towns 1986) 20 football teams take part in a tournament. On the first day all the teams play one match. On the second day all the teams play a further match. Prove that after the second day it is possible to select 10 teams, so that no two of them have yet played each other.

   Solution: This is a graph on 20 vertices whose edge set is a union of 2 matchings. So all degrees ≤ 2, and graph is a disjoint union of cycles and paths. Clearly possible to get independent set of size half.

2. (BAMO 2004/3) NASA has proposed populating Mars with 2,004 settlements. The only way to get from one settlement to another will be by a connecting tunnel. A bored bureaucrat draws on a map of Mars, randomly placing N tunnels connecting the settlements in such a way that no two settlements have more than one tunnel connecting them. What is the smallest value of N that guarantees that, no matter how the tunnels are drawn, it will be possible to travel between any two settlements?

   Solution: This is asking for the max number of edges in a disconnected graph, after which we add 1. Suppoose we have a maximal disconnected graph. Then each connected component must be a clique. By convexity, the answer corresponds to a single K_{2003} plus an isolated vertex.

3. (BAMO 2005/4) There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.

   Solution: The key is that 1000 is even. Reduce to the case when the graph is a spanning tree. Take a leaf and its neighbor, pave the edge between them, and then delete both. Repeating 500 times, we will find a 1-factor.

4. (St. Petersburg 1996/4) In a group of several people, some are acquainted with each other and some are not. Every evening, one person invites all of his acquaintances to a party and introduces them to each other. Suppose that after each person has arranged at least one party, some two people are still unacquainted. Prove that they will not be introduced at the next party.

   Solution: We prove the stronger statement that at the end of this process, all connected components are cliques. For this, it suffices to show that if the initial graph was connected, then the final graph is a clique. Use induction: let v be the last guy, and let G' be the acquaintance graph right before his party. Then $G' - v$ is a disjoint union of cliques, and each clique is connected back to v. So when v holds his party, everything connects into a big clique.

5. (Czech-Slovak Match 1997/2) In a community of more than six people, each member exchanges letters with precisely three other members of the community. Prove that the community can be divided into two nonempty groups so that each member exchanges letters with at least two members of the group he belongs to.

   Solution: Consider a shortest cycle, and let that be one of the groups. Put everybody else in the other group, and then do small alterations if necessary.

3.2 Extremal graph theory

> *Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.*
>
> (The opening sentence in *Extremal Graph Theory*, by Béla Bollobás.)

This very interesting field happens to be the subject of my own research, as well as one of the most common sources of advanced graph theory problems in Olympiads. The most famous theorems concern what substructures can be forced to exist in a graph simply by controlling the total number of edges. The classical starting point is Turán's theorem, which proves the extremality of the following graph: let $T_r(n)$ be the complete r-partite graph with its n vertices distributed among its r parts as evenly as possible (because rounding errors may occur).

Theorem. *(Turán) For $r \geq 3$, the Turán graph $T_{r-1}(n)$ is the unique n-vertex graph with the maximum number of edges subject to having no K_r subgraphs.*

An excellent proof of Turán's theorem can be found on page 167 of the book *Graph Theory*, by Reinhard Diestel. This is a well-written book which has an electronic edition **freely available** on the author's website!

Another fundamental question is to ask how many subgraphs of a certain type must be forced by a given number of edges. The following results can be proven by simply letting d_i denote the degree of the i-th vertex, and using the inequality[1] that $\sum_{i=1}^n \binom{d_i}{r} \geq n \cdot \binom{\overline{d}}{r}$, where $\overline{d}$ is the average value of d_i.

1. Any n-vertex graph with m edges must have $\geq n \cdot \binom{2m/n}{2}$ subgraphs isomorphic to the 3-vertex "V-shape" $K_{1,2}$. Here, we extend the $\binom{x}{r}$ notation to allow non-integral values of x, by defining it to be $\frac{1}{r!} \cdot x(x-1)\cdots(x-r+1)$.

2. More generally, for any integer $r \geq 2$, any n-vertex graph with m edges must have $\geq n \cdot \binom{2m/n}{r}$ subgraphs isomorphic to the r-pointed "star" $K_{1,r}$.

Now try these problems.

1. Prove that the number of edges in the Turán graph $T_r(n)$ is less than or equal to $\left(1 - \frac{1}{r}\right)\frac{n^2}{2}$, with equality when r divides n.

   Solution: Let x_i be the part sizes. Then $\sum x_i = n$, and the number of edges is $\sum_{i<j} x_i x_j = \frac{1}{2}\left[\left(\sum x_i\right)^2 - \sum x_i^2\right]$. By convexity, $\sum x_i^2 \geq r \cdot \left(\frac{n}{r}\right)^2 = \frac{n^2}{r}$, and we just plug that in.

2. Prove that the Turán graph $T_r(n)$ is the r-colorable graph with the maximum number of edges.

   Solution: Clearly Turán graph is r-colorable. But any graph with more edges contains a K_{r+1}, so that portion cannot be r-colored, hence the entire graph cannot be r-colored.

3. (approximation to Erdős-Sós conjecture, does not use Turán's Theorem) If a graph G has average degree d, then it has a subgraph $H \subset G$ whose minimum degree is $\geq d/2$. **Corollary:** for any tree T, if a graph G has average degree $\geq 2|T|$, then G contains T as a subgraph.

   Solution: The condition that average degree $\geq d$ is equivalent to the number of edges being $\geq nd/2$. Now keep deleting vertices of degree $< d/2$. This will preserve the above condition, but must terminate because the average degree never goes below d, so in particular the number of vertices cannot go below d. Thus, we will stop at a graph with all vertices of degree $\geq d/2$. Then embed the tree greedily.

4. (Japan 1998/2) A country has 1998 airports connected by some direct flights. For any three airports, some two are not connected by a direct flight. What is the maximum number of direct flights that can be offered?

   Solution: By Turán, the largest triangle-free subgraph of K_{1998} is bipartite with sides of size 999, so 999^2 is the answer.

5. (Japan 1997/3) Let G be a graph with 9 vertices. Suppose given any five points of G, there exist at least 2 edges with both endpoints among the five points. What is the minimum possible number of edges in G?

   Solution: The optimal configuration is a union of 3 disjoint triangles. But more work is required to prove that it is actually optimal.

3.3 Matching

Consider a bipartite graph $G = (V, E)$ with partition $V = A \cup B$. A *matching* is a collection of edges which have no endpoints in common. We say that *A has a perfect matching to B* if there is a matching which hits every vertex in A.

[1]This inequality will be proved in my Convexity lecture.

Theorem. *(Hall's Marriage Theorem) For any set $S \subset A$, let $N(S)$ denote the set of vertices (necessarily in B) which are adjacent to at least one vertex in S. Then, A has a perfect matching to B if and only if $|N(S)| \geq |S|$ for* ***every*** *$S \subset A$.*

This has traditionally been called the "marriage" theorem because of the possible interpretation of edges as "acceptable" pairings, with the objective of maximizing the number of pairings. In real life, however, perhaps there may be varying degrees of "acceptability." This may be formalized by giving each vertex (in both parts) an ordering of its incident edges. Then, a matching M is called *unstable* if there is an edge $e = ab \notin M$ for which both a and b both prefer the edge e to their current partner (according to M).

Theorem. *(Stable Marriage Theorem) A stable matching always exists, for every bipartite graph and every collection of preference orderings.*

Now try these problems.

1. Let A be a square $n \times n$ matrix of nonnegative integers, in which each row and column sum up to the positive integer m. Prove that A can be expressed as a sum of m *permutation matrices* $A = P_1 + \cdots P_m$. Here, a permutation matrix is an $n \times n$ matrix of zeros and ones, such that each row contains a single "1", and each column contains a single "1."

   Solution: Pull out permutation matrices inductively via Hall's theorem. LHS is the rows, RHS is the columns, and there is an edge between i on the LHS and j on the RHS if a_{ij}, the (i, j) entry of A, is > 0. Actually, write the number a_{ij} next to that edge.

   For any subset S of the LHS, note that the sum of all numbers on edges coming out of S is precisely $m|S|$. But those edges are a subset of the edges coming out of $N(S)$, so their sum is $\leq m|N(S)|$. Thus $|N(S)| \geq |S|$ for any S.

2. Find a graph for which the largest stable matching does not have the maximum cardinality among all not-necessarily-stable matchings.

   Solution: Let the LHS be $\{a, b, c\}$, and the RHS be $\{1, 2, 3\}$. There are 6 edges forming a C_6, going cyclically $a1b2c3$. Make some preferences as follows: 1 prefers b over a, b prefers 1 over 2, c prefers 3 over 2, and 3 prefers c over a. The other preferences can be arbitrary. There are only two maximal ordinary matchings because the graph is a C_6, and both of them are unstable. The new potential edge culprits in each case are the edge $b1$ and the edge $c3$.

3. (Turkey 1998/4) To n people are to be assigned n different houses. Each person ranks the houses in some order (with no ties). After the assignment is made, it is observed that every other assignment assigns at least one person to a house that person ranked lower than in the given assignment. Prove that at least one person received his/her top choice in the given assignment.

   Solution: Use existence of a stable marriage. Then take a person, and consider swapping him up to his top choice. Since things are finite, eventually we loop, and then we can improve them all.

4. (Diestel 2.7) Let $S = \{1, 2, \ldots, kn\}$, and suppose $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are both partitions of S into n sets of size k. Then there exists a set T of size n such that every intersection $T \cap A_i$ and $T \cap B_i$ has cardinality exactly 1.

   Solution: Let the LHS have n vertices corresponding to the A_j, and the RHS have n vertices corresponding to the B_j. Consider any $i \in S$. Since we have two partitions, i is in exactly one A-set and exactly one B-set. Say they were A_a and B_b. Then, put an edge between the corresponding vertices in the graph, and label it with i.

   Note that any perfect matching will give a set T: just let T be the collection of the labels of the edges in the matching. To see that it exists, consider any subset S of LHS, and note that since all sets have size exactly k, the number of edges coming out of S is precisely $k|S|$. But this is a subset of the edges coming out of $N(S)$, so $k|S| \leq k|N(S)|$, done.

5. (Diestel 2.8) Let A be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n \in \mathbb{N}$. Show that there are disjoint subsets $D_k \subset A_k$ with $|D_k| = d_k$ for all k, if and only if

$$\left|\bigcup_{i \in I} A_i\right| \geq \sum_{i \in I} d_i,$$

for all $I \subset \{1, \ldots, n\}$.

3.4 Ramsey theory

Complete disorder is impossible.

— T. S. Motzkin, on the theme of Ramsey Theory.

Let s and t be positive integers. We define the *Ramsey Number* $R(s,t)$ to be the minimum integer n for which **every** red-blue coloring of the edges of K_n contains either a completely red K_s or a completely blue K_t. Ramsey's Theorem states that $R(s,t)$ is always finite, and we will prove this in the first exercise below. The interesting question in this field is to find upper and lower bounds for these numbers, as well as for quantities defined in a similar spirit.

Now try these problems.

1. Prove by induction that $R(s,t) \leq \binom{s+t-2}{s-1}$. Note that in particular, $R(3,3) \leq 6$.

   Solution: Observe that $R(s,t) \leq R(s-1,t) + R(s,t-1)$, because if we have that many vertices, then if we select one vertex, then it cannot simultaneously have $< R(s-1,t)$ red neighbors and $< R(s,t-1)$ blue neighbors, so we can inductively build either a red K_s or a blue K_t. But

   $$\binom{(s-1)+t-2}{(s-2)} + \binom{s+(t-1)-2}{s-1} = \binom{s+t-2}{s-1},$$

   because in Pascal's Triangle the sum of two adjacent guys in a row equals the guy directly below them in the next row.

2. (IMO 1964/4) Seventeen people correspond by mail with one another-each one with all the rest. In their letters only three different topics are discussed. each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

   Solution: This is asking us to prove that the 3-color Ramsey Number $R(3,3,3)$ is ≤ 17. By the same observation as in the previous problem, $R(a,b,c) \leq R(a-1,b,c) + R(a,b-1,c) + R(a,b,c-1) - 1$. Then using symmetry, $R(3,3,3) \leq 3R(3,3,2) - 1$. It suffices to show that $R(3,3,2) \leq 6$. But this is immediate, because if we have 6 vertices, if we even use the 3rd color on a single edge, we already get a K_2. So we cannot use the 3rd color. But then from above, we know $R(3,3) \leq 6$, so we are done.

3. (South Africa 1997/5) Six points are joined pairwise by red or blue segments. Must there exist a closed path consisting of four of the segments, all of the same color?

   Solution: Yes. Proof: assume not. Let the vertices be a, b, c, d, e, f. Use Ramsey to get a monochromatic triangle first, and suppose it is a, b, c. WLOG it is blue. Now d cannot have 2 blue edges into a, b, c, or else we get blue C_4. Same for e and f. Also, if, say, a, b both have red edges to each of say, d, e, then we get C_4. Therefore, the only possible configuration is to have a blue matching between $\{a, b, c\}$ and $\{d, e, f\}$, and all other edges between those sets are red. WLOG the blue matching is ad, be, cf. But then edges de and ef are both forced red, or else we have blue C_4, say $abed$. And then there is a red C_4: $bfed$.

4. (IMO 1978/6) The members of an international society belong to 6 different countries (each to only one country). The list of the members contain 1978 names numbered 1, 2, ..., 1978. Prove that there exists at least one member whose number is the sum of the numbers of two of his compatriots, not necessarily distinct.

(The IMO problems were listed in the article *Ramsey Theory and the IMO*, by Cambridge professor Ben Green.)

3.5 Planarity

When we represent graphs by drawing them in the plane, we draw edges as curves, permitting intersections. If a graph has the property that it can be drawn in the plane without any intersecting edges, then it is called *planar*. Here is the tip of the iceberg. Perhaps the most useful planarity theorem in Olympiad problems is the Euler Formula. (The second result is less useful for Olympiads, but is too famous to omit.)

- (Euler Formula) Every connected planar graph satisfies $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces.
- (Four-color theorem) Every planar graph is 4-colorable.

Now use the Euler Formula to solve these problems.

1. If G is a connected, planar, simple graph, then $E \le 3V - 6$.

   Solution: For each face, calculate its perimeter, and add all of these up. This double-counts each edge. Each face has perimeter ≥ 3, so we get $2E \ge 3F$. Plugging in, we have
   $$2 = V - E + F \le V - E + \frac{2}{3}E = V - \frac{1}{3}E.$$

2. Show that K_5 is not planar.

   Solution: $V = 5$, $E = 10$, so we must have $F = 2 - V + E = 7$. But as in the previous solution, we need to have $2E \ge 3F$, which is not the case.

3. Show that $K_{3,3}$ is not planar.

   Solution: $V = 6$, $E = 9$, so we must have $F = 2 - V + E = 5$. But as in the previous solution, we need to have $2E \ge 3F$. Actually, we need $2E \ge 4F$, because $K_{3,3}$ has no triangles. But this stronger inequality is false.

4 Problems

1. (IMO Shortlist 2004/C3) The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \ge 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from a complete graph on n vertices (where each pair of vertices are joined by an edge).

   Solution: First we show that we cannot end up with any graph with $\le n - 1$ edges. We are only breaking cycles, so we cannot destroy connectivity. Therefore, any final graph with $\le n - 1$ edges must have exactly $n - 1$ edges, and be a tree, hence bipartite. But if we consider the reverse process, observe that if we start from a bipartite graph and complete C_4's, we will stay bipartite, and K_n is not bipartite!

   It remains to find an n-edge graph that we can reach. I think one such graph is a triangle plus a single path leading out of one of the vertices of the triangle. See if you can prove this.

2. (USAMO 1989/2) The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

   Solution: Take a maximal matching. Suppose it has $t \le 5$ edges. Then there are $20 - 2t$ vertices not in this matching, and since matching is maximal, those vertices span no edges. But every vertex has degree ≥ 1, so we must have $\ge 20 - 2t$ edges from those leftover vertices. This requires a total of $\ge (20 - 2t) + t = 20 - t \ge 15$ edges, which is too many.

3. (Russia 1998/48) There are 1998 cities in Russia, each being connected (in both directions) by flights to three other cities. Any city can be reached by any other city by a sequence of flights. The KGB plans to close off 200 cities, no two joined by a single flight. Show that this can be done so that any open city can be reached from any other open city by a sequence of flights only passing through open cities.

4. (St. Petersburg 1998/17) A regiment consists of 169 men. Each day, four of them are on duty. Is it possible that at some point, any two men have served together exactly once?

5. (Vietnam 1997/1) Determine the smallest integer k for which there exists a graph on 25 vertices such that every vertex is adjacent to exactly k others, and any two nonadjacent vertices are both adjacent to some third vertex.

6. (St. Petersburg 1997/14) In a federation consisting of two republics, each pair of cities is linked by a one-way road, and each city can be reached from each other city by these roads. The Hamilton travel agency provides n different tours of the cities of the first republic (visiting each city once and returning to the starting city without leaving the republic) and m tours of the second republic. Prove that Hamilton can offer mn such tours around the whole federation.

7. (Spain 1996/5) At Port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered.

5 Harder problems

1. (St. Petersburg 1996/24) There are 2000 towns in a country, each pair of which is linked by a road. The Ministry of Reconstruction proposed all of the possible assignments of one-way traffic to each road. The Ministry of Transportation rejected each assignment that did not allow travel from any town to any other town. Prove that more of half of the assignments remained.

2. (IMO 2007/3) In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its size. Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

6 Real problems

These are some of the most famous open problems in graph theory. Attempts to solve them have led to the discovery of many interesting and useful other results. They are probably very hard, but they are an example of the happy fact that combinatorialists often can describe the problems they are working on to normal people!

1. (List coloring conjecture) Let $\chi'(G)$ be the minimum number of colors required to color the edges of G such that incident edges receive different colors. Now, consider a slightly different notion of coloring, where instead of using the same pool of colors for each edge, we give each edge an individual list of colors (possibly drawn from a much larger pool) that it can pick from. The parameter $\text{ch}'(G)$ is then defined as the minimum integer k such that no matter how we assign lists of size k to the edges of G, it is always possible to color the edges from the lists in such a way that incident edges receive different colors. Prove that $\chi'(G) = \text{ch}'(G)$ for all graphs G.

2. (Hadwiger's conjecture) If a graph has chromatic number at least r, then it contains a K_r-minor. That is, we can find disjoint subsets $V_1, \ldots, V_r$ of the vertex set such that:

   - every induced subgraph $G[V_i]$ is a connected graph, and
   - for every $i < j$, we can find an edge that has one endpoint in V_i and one endpoint in V_j.

7 Really harder problems

Determine the exact value of the Ramsey number $R(5,5)$. Hint: it is known to be one of $\{43, 44, 45, 46, 47, 48, 49\}$. You may use as many supercomputers as you want.

Believe it or not, this is unknown. For a greater challenge, determine $R(6,6)$. If you succeed, Paul Erdős would have been proud.

> *Imagine an alien force, vastly more powerful than us landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they asked for $R(6,6)$, we should attempt to destroy the aliens.*
>
> -Paul Erdős