Convexity

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1 Warm-up

Sharpening inequalities can help to reduce the number of variables. **Prove that inequality (2) is sharper than (1),** for all reals $x, y \geq 0$.

$$\frac{(x+y)^2}{16} + 1 \geq \sqrt{xy} \tag{1}$$

$$\frac{(x+y)^2}{16} + 1 \geq \frac{x+y}{2} \tag{2}$$

Then prove (2), using the substitution $t = \frac{x+y}{2}$. Single-variable inequalities can be quite easy!

Solution: The substitution gives something equivalent to $(\frac{t}{2} - 1)^2 \geq 0$.

2 Tools

Recall that a set S (say in the plane) is *convex* if for any $x, y \in S$, the line segment with endpoints x and y is completely contained in S. It turns out that this concept is very useful in the theory of functions.

Definition. *We say that a function $f(x)$ is* ***convex*** *on the interval I when the set $\{(x, y) : x \in I, y \geq f(x)\}$ is convex. On the other hand, if the set $\{(x, y) : x \in I, y \leq f(x)\}$ is convex, then we say that f is* ***concave****. Note that it is possible for f to be neither convex nor concave. We say that the convexity/concavity is* ***strict*** *if the graph of $f(x)$ over the interval I contains no straight line segments.*

Remark. Plugging in the definition of set-theoretic convexity, we find the following equivalent definition. The function f is convex on the interval I iff for every $a, b \in I$, the line segment between the points $(a, f(a))$ and $(b, f(b))$ is always above or on the curve f. Analogously, f is concave iff the line segment always lies below or on the curve. This definition is illustrated in Figure 1.

After the above remark, the following famous and useful inequality should be quite believable and easy to remember.

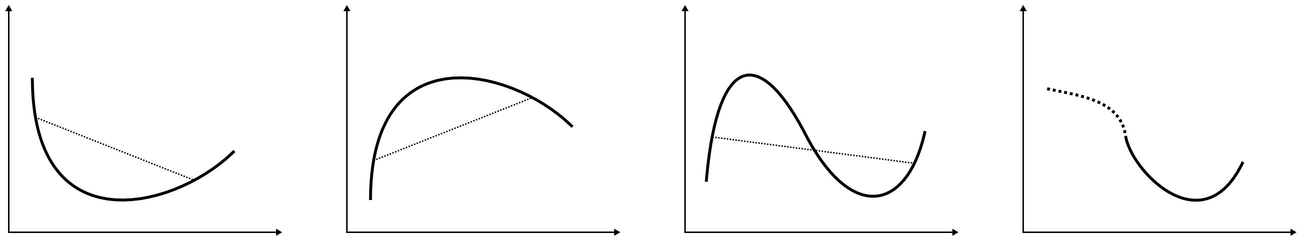
Theorem. *(Basic version of Jensen's Inequality) Let $f(x)$ be* ***convex*** *on the interval I. Then for any $a, b \in I$,*

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2}.$$

On the other hand, if $f(x)$ is ***concave*** *on I, then we have the reverse inequality for all $a, b \in I$:*

$$f\left(\frac{a+b}{2}\right) \geq \frac{f(a) + f(b)}{2}.$$

Remark. This can be interpreted as "for convex f, the average of f exceeds f of the average," which motivates the general form of Jensen's inequality:



(i) (ii) (iii) (iv)

Figure 1: The function in (i) is convex, (ii) is concave, and (iii) is neither. In each diagram, the dotted line segments represent a sample line segment as in the definition of convexity. However, note that a function that fails to be globally convex/concave can be convex/concave on parts of their domains. For example, the function in (iv) is convex on the part where it is solid and concave on the part where it is dotted.

Theorem. *(Jensen's Inequality) Let $f(x)$ be **convex** on the interval I. Then for any $x_1, \ldots, x_t \in I$,*

$$f(\textit{average of } \{x_i\}) \leq \textit{average of } \{f(x_i)\}.$$

*If f were **concave** instead, then the inequality would be reversed.*

Remark. We wrote the above inequality without explicitly stating what "average" meant. This is to allow weighted averages, subject to the condition that the same weight pattern is used on both the LHS and RHS.

2.1 That's great, but how do I prove that a function is convex?

1. If you know calculus, take the second derivative. It is a well-known fact that if the second derivative $f''(x)$ is ≥ 0 for all x in an interval I, then f is **convex** on I. On the other hand, if $f(x) \leq 0$ for all $x \in I$, then f is **concave** on I.

2. By the above test or by inspection, here are some basic functions that you should safely be able to claim are convex/concave.

   - Constant functions $f(x) = c$ are both **convex** and **concave**.
   - Powers of x: $f(x) = x^r$ with $r \geq 1$ are **convex** *on the interval* $0 < x < \infty$, and with $0 < r \leq 1$ are **concave** on that same interval. (Note that $f(x) = x$ is both convex and concave!)
   - Reciprocal powers: $f(x) = \frac{1}{x^r}$ are **convex** *on the interval* $0 < x < \infty$ for all powers $r > 0$. For negative odd integers r, $f(x)$ is **concave** *on the interval* $-\infty < x < 0$, and for negative even integers r, $f(x)$ is **convex** *on the interval* $-\infty < x < 0$.
   - The logarithm $f(x) = \log x$ is **concave** *on the interval* $0 < x < \infty$, and the exponential $f(x) = e^x$ is **convex** everywhere.

3. $f(x)$ is convex iff $-f(x)$ is concave.

4. You can combine basic convex functions to build more complicated convex functions.

   - If $f(x)$ is convex, then $g(x) = c \cdot f(x)$ is also convex for any *positive* constant multiplier c.
   - If $f(x)$ is convex, then $g(x) = f(ax + b)$ is also convex for *any* constants $a, b \in \mathbb{R}$. But the interval of convexity will change: for example, if $f(x)$ were convex on $0 < x < 1$ and we had $a = 5, b = 2$, then $g(x)$ would be convex on $2 < x < 7$.
   - If $f(x)$ and $g(x)$ are convex, then their sum $h(x) = f(x) + g(x)$ is convex.

5. If you are brave, you can invoke the oft-used claim "by observation," e.g,

$$f(x) = |x| \textit{ is convex by observation.}$$

The above statement might actually pass, depending on the grader, but the following desparate statement definitely will not:

$$f(x) = \tfrac{(x+1)^2}{2x^2+(3-x)^2} \textit{ is concave by observation.}$$

(It is false anyway, so that would ruin your credibility...)

Now prove the following results.

1. $f(x) = \frac{1}{1-x}$ is convex on $-\infty < x < 1$.
2. For any constant $c \in \mathbb{R}$, $f(x) = \frac{x^2}{c-x}$ is convex on $-\infty < x < c$.
   Solution:
   $$\frac{x^2}{c-x} = -\frac{x^2}{x-c} = -\left[x + c + \frac{c^2}{x-c}\right] = -x - c - \frac{c^2}{x-c}$$
   Each summand is convex.
3. $f(x) = \frac{x(x-1)}{2}$ is convex.
   Solution: Expand: $x^2/2 - x/2$. This is sum of two convex functions.
4. $f(x) = \frac{x(x-1)\cdots(x-r+1)}{r!}$ is convex on $r - 1 < x < \infty$.
   Solution: Forget about the $r!$ factor. Think of $f(x)$ as a product of r linear terms. Then by the product rule, f' is the sum of r products, each consisting of $r-1$ linear terms. For example, one of the terms would be $x(x-1)(x-3)(x-4)\cdots(x-r+1)$; this corresponds to the $x-2$ factor missing. But then the second derivative f'' is a sum of even more products, but each product consists of $r-2$ linear terms (with two factors missing). But for $x > r$, every factor is > 0, and we are taking a sum of products of them, so $f'' > 0$.

2.2 Direct applications of Jensen

Now use direct applications of Jensen's inequality to prove the following inequalities.

1. (India 1995, from Kiran) Let $x_1, \ldots, x_n$ be positive numbers summing to 1. Prove that
   $$\frac{x_1}{\sqrt{1-x_1}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}.$$
   Solution: Done immediately by Jensen, just need to prove that $x/\sqrt{1-x}$ is convex on the interval $0 < x < 1$. Use the substitution $t = 1 - x$, and prove convexity (on the same interval) of $(1-t)/\sqrt{t} = 1/\sqrt{t} + (-\sqrt{t})$. But this is the sum of two convex functions, hence convex!
2. (MMO[†] 1963) For $a, b, c > 0$, prove:
   $$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$
   Solution: Homogeneous, so WLOG $a + b + c = 1$. Then LHS is sum over terms of the form $x/(1-x) = -1 + 1/(1-x)$, which is convex on $0 < x < 1$. So by Jensen, it is $\geq$ the case when everything is at the average $1/3$. This gives $3/2$.

[†]not the "Massively Multiplayer" Olympiad

3. Let a, b, c be positive real numbers with $a+b+c \geq 1$. Prove that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{1}{2}.$$

Solution: Let $S = a+b+c$. Then the fractions are of the form $\frac{x^2}{S-x}$, which is convex by the exercise in the previous section. Hence LHS is $\geq 3 \cdot \frac{(S/3)^2}{(2/3)S} = S/2$, and we assumed that $S \geq 1$.

4. (Ireland 1998/7a) Prove for all positive real numbers a, b, c:

$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right).$$

Solution: Use substitution $S = a+b+c$. WLOG, $S = 1$. Then we need convexity of $1/(1-x)$ on $(0,1)$, which is clear.

2.3 Endpoints of convex functions

- If $f(x)$ is **convex** on the interval $a \leq x \leq b$, then $f(x)$ attains a maximum, and that value is either $f(a)$ or $f(b)$.
- If $f(x)$ is **concave** on the interval $a \leq x \leq b$, then $f(x)$ attains a minimum, and that value is either $f(a)$ or $f(b)$.

Now try these problems. Use the above facts to observe that it suffices to manually check just a few possible such assignments. Finally, check those values!

1. (Bulgaria, 1995) Let $n \geq 2$ and $0 \leq x_i \leq 1$ for all $i = 1, 2, \ldots, n$. Show that

$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots x_nx_1) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Bonus: determine when there is equality.

Solution: This is actually linear in each variable separately. Therefore, we can iteratively unsmooth each variable to an endpoint $\{0, 1\}$. Rewrite LHS as

$$x_1(1-x_2) + x_2(1-x_3) + \cdots x_n(1-x_1).$$

With $\{0,1\}$-assignments, each term is zero unless $x_i = 1$ and $x_{i+1} = 0$, in which case the term is 1. This is a "1, 0" pattern in consecutive variables when reading the x_i cyclically. Clearly, the maximum possible number of "1, 0" patterns is the claimed RHS.

2. (USAMO 1980/5) Show that for all real numbers $0 \leq a, b, c \leq 1$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Solution: Consider b, c as constant, and see what happens if we perturb a. Then, the LHS is a convex function of a, because first term is linear, 2nd/3rd terms are of the form $\frac{\cdot}{x+\cdot}$, and the third term is a purely linear function of a. Hence LHS is maximized at one of the extreme values of a (either 0 or 1), so we may assume it is one of these. Similarly, $b, c \in \{0,1\}$. This gives 8 assignmets of (a, b, c) to check, and they all work.

2.4 Smoothing and unsmoothing

Theorem. *Let $f(x)$ be* ***convex*** *on the interval I. Suppose $a < b$ are both in I, and suppose $\epsilon > 0$ is a real number for which $a + \epsilon \le b - \epsilon$. Then $f(a) + f(b) \ge f(a + \epsilon) + f(b - \epsilon)$. If f is* ***strictly convex****, then the inequality is strict.*

This means that:

- For **convex** functions f, we can decrease the sum $f(a) + f(b)$ by "smoothing" a and b together, and increase the sum by "unsmoothing" a and b apart.
- For **concave** functions f, we can increase the sum $f(a) + f(b)$ by "smoothing" a and b together, and decrease the sum by "unsmoothing" a and b apart.
- In all of the above statements, if the convexity/concavity is **strict**, then the increasing/decreasing is strict as well.

This "smoothing principle" gives another way to draw conclusions about the assignments to the variables which bring the LHS and RHS closest together (i.e., sharpening the inequality). Hopefully, this process will give us a simpler inequality to prove.

Warning. Be careful to ensure that your smoothing process terminates. For example, if we are trying to prove that $(a + b + c)/3 \ge \sqrt[3]{abc}$ for $a, b, c \ge 0$, we can observe that if we smooth any pair of variables together, then the LHS remains constant while the RHS increases. Therefore, a naïve smoothing procedure would be:

> *As long as there are two unequal variables, smooth them both together into their arithmetic mean.*
> *At the end, we will have $a = b = c$, and the RHS will be exactly equal to the LHS, so we are done.*

Unfortunately, "the end" may take infinitely long to occur, if the initial values of a, b, c are unfavorable! Instead, one could use a smoothing argument for which each iteration increases the number of a, b, c equal to their arithmetic mean $(a + b + c)/3$.

Now try these problems.

1. (Zvezda 1998) Prove for all reals $a, b, c \ge 0$:

$$\frac{(a+b+c)^2}{3} \ge a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}.$$

   Solution: If we smooth, say, a and b together, then LHS is invariant, and RHS is of the form $c\sqrt{ab} + \sqrt{c}\sqrt{ab}(\sqrt{a} + \sqrt{b})$. But $\sqrt{ab}$ will grow, as will $\sqrt{a} + \sqrt{b}$. Smooth until each element hits the arithmetic mean. Therefore, RHS will grow, so we may assume that all are equal. This corresponds to $(3t)^2/3 \ge 3t^2$, which is indeed true.

3 Problems

Now you're on your own. Use any method from this lecture to solve these problems.

1. Let $m \ge n$ be positive integers. Prove that every graph with m edges and n vertices has at least $\frac{m^2}{n}$ "V-shapes," which are defined to be unordered triples of vertices which have exactly two edges between themselves.

   Solution: The number is exactly $\sum \binom{d_i}{2}$, where the d_i are the degrees. Average degree is $2m/n$, and $\binom{x}{2}$ is convex, so Jensen implies that this is $\ge n\binom{2m/n}{2} \ge n(2m/n)(2m/n - 1)/2 \ge m^2/n$.

2. (Ireland 1998/7b) Prove that if a, b, c are positive real numbers, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \le \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Solution: Jensen with $f(t) = 1/t$:

$$\frac{f(a) + f(b)}{2} \ge f\left(\frac{a+b}{2}\right).$$

3. (T. Mildorf's *Inequalities*, problem 3) Let $a_1, \ldots, a_n \ge 0$ be real numbers summing to 1. Prove that

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n \le \frac{1}{4}.$$

Solution: Observe that if any of the a_i are zero, then we could increase the LHS by deleting the zero entries (thus reducing the value of n). For example, if $n = 4$ and the sequence of a_i's was $(0.5, 0, 0.4, 0.1)$, then the corresponding LHS with $n = 3$ and a sequence of $(0.5, 0.4, 0.1)$ would be higher. Therefore, we may assume that the sequence of a_i has no zeros.

Next, consider unsmoothing the pair corresponding to a_1 and a_3. Observe that a_1 is only multiplied by a_2, but a_3 is multiplied by $a_2 + a_4 > a_2$ if $n \ge 4$, because we just showed all $a_i > 0$. Therefore, we may increase the LHS by pushing a_1 all the way to zero, and giving its mass to a_3. Repeat this process (at most $n < \infty$ times) until $n \le 3$.

Finally, since we only have a_1, a_2, and a_3 left, the LHS is simply equal to $a_2(a_1 + a_3) = a_2(1 - a_2)$ because they sum to 1. Clearly, this has maximum value 1/4. (There are also two other cases which correspond to ending up at $n = 2$ or $n = 1$, but those are trivial.)

4. (Hong Kong 2000) Let $a_1 \le \cdots \le a_n$ be real numbers such that $a_1 + \cdots + a_n = 0$. Show that

$$a_1^2 + \cdots + a_n^2 + na_1a_n \le 0.$$

Solution: If there are at least two intermediate a_i, neither of which are equal to a_1 or a_n, then we can unsmooth them and increase LHS. So we may assume that there are x of the a_i equal to some $a = a_1$, at most one of the a_i equal to some b, and $n - x$ or $n - 1 - x$ of the a_i equal to some c.

Case 1 (when there is no b): using $xa + (n - x)c = 0$, we solve and get $x = nc/(c - a)$. Then $xa^2 + (n - x)c^2 = -nac$, exactly what we needed.

Case 2 (when there is one b): using $xa + b + (n - 1 - x)c = 0$, we solve and get $x = [b + (n - 1)c]/(c - a)$. Then $xa^2 + b^2 + (n - 1 - x)c^2 = -(n - 1)ac + b(-a - c) + b^2$. It suffices to show that $b(-a - c) + b^2 \le -ac$. If $b \ge 0$, then use $b \le c$ to get $b^2 \le bc$, hence $b(-a - c) + b^2 \le -ab \le -ac$, final inequality using that $a \le 0$ and $b \le c$. On the other hand, if $b \le 0$, then use $b \ge a$ to get $b^2 \le ab$, hence $b(-a - c) + b^2 \le -bc \le -ac$, final inequality using that $c \ge 0$ and $b \ge a$.

5. (IMO 1984) For $x, y, z > 0$ and $x + y + z = 1$, prove that $xy + yz + xz - 2xyz \le 7/27$.

Solution: Smooth with the following expression: $x(y + z) + yz(1 - 2x)$. Now, if $x \le 1/2$, then we can push y and z together. The mushing algorithm is as follows: first, if there is one of them that is greater than 1/2, pick any other one and mush the other two until all are within 1/2. Next we will be allowed to mush with any variable taking the place of x. Pick the middle term to be x; then by contradiction, the other two terms must be on opposite sides of 1/3. Hence we can mush to get one of them to be 1/3. Finally, use the 1/3 for x and mush the other two into 1/3. Plugging in, we get 7/27.

6. (MOP 2008/6) Prove for real numbers $x \ge y \ge 1$:

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+1}} + \frac{1}{\sqrt{x+1}} \ge \frac{y}{\sqrt{x+y}} + \frac{x}{\sqrt{x+1}} + \frac{1}{\sqrt{y+1}}.$$

7. (MOP Team Contest 2008) Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $\prod x_i = 1$. Prove:

$$\sum \frac{1}{n-1+x_i} \le 1.$$

8. (MOP 1998/5/5) Let $a_1 \ge \cdots \ge a_n \ge a_{n+1} = 0$ be a sequence of real numbers. Prove that:

$$\sqrt{\sum_{k=1}^{n} a_k} \le \sum_{k=1}^{n} \sqrt{k}(\sqrt{a_k} - \sqrt{a_{k+1}}).$$

Solution: Since the inequality is homogeneous, we can normalize the a_k so that $a_1 = 1$. (If they are all zero, it is trivial anyway.) Now define the random variable X such that $P(X \ge k) = \sqrt{a_k}$. Then STS

$$\sqrt{\mathbb{E}[\min\{X_1, X_2\}]} \le \mathbb{E}[\sqrt{X}],$$

where X, X_1, X_2 are i.i.d. Prove by induction on n. Base case is if $n = 1$, trivial. Now if you go to $n+1$ by shifting q amount of probability from $P(X = n)$ to $P(X = n+1)$, RHS will increase by exactly $q(\sqrt{n+1} - \sqrt{n})$. Yet LHS increases by exactly q^2 under the square root. Now since the probability shifted from $P(X = n)$, the square root was originally at least $q^2 n$. In the worst case, the LHS increases by $\sqrt{q^2 n + q^2} - \sqrt{q^2 n}$, which equals the RHS increase.

A Compactness

Definition. *Let $f(x_1, \ldots, x_t)$ be a function with domain D. We say that the function* ***attains a maximum on*** *D at some assignment $(x_i) = (c_i)$ if $f(c_1, \ldots, c_t)$ is greater than or equal to every other value $f(x_1, \ldots, x_t)$ with $(x_1, \ldots, x_t) \in D$.*

Remark. Not every function has a maximum! Consider, for example, the function $1/x$ on the domain $0 < x < \infty$, or even the function x on the domain $0 < x < 1$.

Definition. *Let D be a subset of $\mathbb{R}^n$.*

- *If D "includes its boundary", then we say that D is* ***closed****.*
- *If there is some finite radius r for which D is contained within the ball of radius r around the origin, then we say that D is* ***bounded****.*
- *If D is both closed and bounded, then[1] we say D is* ***compact****.*

Theorem. *Let f be a* ***continuous*** *function defined over a domain D which is* ***compact****. Then f attains a maximum on D, and also attains a minimum on D.*

[1] Strictly speaking, this is not the proper definition of "compactness," but rather is a consequence of the Heine-Borel Theorem.