CHM: A.4

Po-Shen Loh

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Problem 1 (A.4) *Call a polynomial* positive reducible *if it can be written as a product of two nonconstant polynomials with positive real coefficients. Let $f(x)$ be a polynomial with $f(0) \neq 0$ such that $f(x^n)$ is positive reducible for some natural number n. Prove that $f(x)$ itself is positive reducible.*

Solution:

Let $a(x) = f(x^n)$. That is, set the coefficient of x^{nk} in $a(x)$ equal to the coefficient of x^k in $f(x^n)$, and set all other coefficients in $a(x)$ to zero. Since a is positive reducible, let it be the product bc. Observe that since a only has coefficients for the x^{kn} monomials, its only nonzero derivatives at zero will be those of the kn-th orders.

Let's start taking derivatives (for brevity, we evaluate them all at zero without writing that out): $0 = a' = b'c + bc'$. Since we were told that $f(0) \neq 0$, it follows that $b(0), c(0)$ are also nonzero. Yet they must be positive. Therefore, b' and c' must both be zero at zero. Do it again: $0 = a'' = b''c + 2b'c' + bc''$. From the previous step, the middle term is zero. And we have the same situation again so $b'' = c'' = 0$ at zero.

Keep going until we hit the n-th derivative. Then $n!a_n = a^{(n)} = b^{(n)}c + b^{(n-1)}c' + \cdots + bc^{(n)}$. Everything in the middle falls out again so we are left with the outer terms; the n-th derivatives of b and c can be nonzero. But all derivatives below that had to be zero from our previous paragraph.

Continue the process. We see that unless we are evaluating a derivative of order divisible by n, the nonzero derivatives are forced to multiply by derivatives that evaluate to zero. It is only when we are divisible by n that they can align to multiply by each other. But for derivatives of order not divisible by n, we know that they must be zero since a only has terms x^{kn}. Therefore, we will find that for b and c, the only nonzero derivatives at zero are of order divisible by n; therefore, b and c are polynomials in x^n as well. Therefore, we can let $y = x^n$ and then have new polynomials $\beta(y)$ and $\gamma(y)$ that are equal to b and c; also, $f(y) = a(x)$, so we find that $f(y) = \beta(y)\gamma(y)$. Also, since we did not perturb the coefficients in going from the latin to the greek, we still have positive reducibility. And we are done.