CHM: A.3

Po-Shen Loh

24 March 2001

Problem 1 (A.3) *Let $R = R_1 \oplus R_2$ be the direct sum of the rings R_1 and R_2, and let N_2 be the annihilator ideal of R_2 (in R_2). Prove that R_1 will be an ideal in every ring $\tilde{R}$ containing R as an ideal if and only if the only homomorphism from R_1 to N_2 is the zero homomorphism.*

Solution:

Clearly, R_1 is closed under addition, so it suffices to show that it is closed under scalar multiplication. Since R is an ideal in $\tilde{R}$, multiplication of any element of R_1 by some $x \in \tilde{R}$ sends it to some element in $R_1 \oplus R_2$. Since we have a direct sum, there exists a well-defined projection homomorphism $\pi : R \to R_2$. Therefore, we can define the family of functions $\phi_x(r_1) = \pi(xr_1)$. We now show that the ϕ_x are homomorphisms as well.

In order for multiplication by x to be defined, we need the ring axioms to hold. In particular, the associative law must be true (we now use ordered-pair notation for elements of our direct sum:

$$\begin{aligned} x \cdot ((r_1, 0)(r_1', r_2')) &= (x \cdot (r_1, 0))(r_1', r_2') \\ x \cdot (r_1 r_1', 0) &= (\alpha, \phi_x(r_1) r_2') \quad \text{for some } \alpha \in R_1 \\ \phi_x(r_1 r_1') &= \phi_x(r_1) r_2'. \end{aligned}$$

The above result must hold for all choices of r_1, r_1', and r_2'. Suppose we choose $r_2' = 0$; then we discover that for all r_1 and r_1', $\phi_x(r_1 r_1') = 0$. Hence for all r_1, r_2', $\phi_x(r_1) r_2' = 0 \Rightarrow \phi_x(r_1) \in N_2$. Now we can show that ϕ_x is a homomorphism.

It is clearly true that $\phi_x(a + b) = \phi_x(a) + \phi_x(b)$, so it suffices to show that $\phi_x(ab) = \phi_x(a)\phi_x(b)$. But from above, we see that $\phi_x(ab) = 0$, and since $\phi_x(a) \in N_2$ and $\phi_x(b) \in R_2$, the RHS is also zero. Therefore we do have a homomorphism, as claimed.

This immediately solves our problem; any $x-$multiplication automatically induces a homomorphism from R_1 to N_2. R_1 is an ideal in the larger ring if and only if all ϕ_x are zero; therefore, we are done.