CHM: A.10

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Problem 1 (A.10, Schweitzer 1967/2) *Let K be a subset of a group G that is not a union of left cosets of a proper subgroup. Prove that if G is a torsion group or if K is a finite set, then the subset*

$$\cap_{k \in K} k^{-1} K$$

consists of the identity alone. [L. Rédei]

Solution:

We first deal with the case $|K| < \infty$. Let $K = \{a_1, a_2, \ldots, a_n\}$. Suppose that there is some element $x \neq 1$ in the intersection; then for each $i \in \{1, 2, \ldots, n\}$, there exists some j for which $a_i^{-1} a_j = x$. Let us renumber the elements of k so that this is true:

$$\begin{aligned} x &= a_1^{-1} a_2 \\ x &= a_2^{-1} a_3 \\ x &= a_3^{-1} a_4 \\ &\cdots \\ x &= a_r^{-1} a_1 \end{aligned}$$

Since we have a finite subset, we are guaranteed to return to a_1 eventually. Multiply these r equations; we find that $x^r = 1$. Since we took r iterations to get back to a_1, we also know that no smaller power of x is the identity. Hence, $|x| = r$.

Let us repeat the process starting with a_{r+1} instead of a_1. Since $|x| = r$, we must have a cycle of r elements. In this way, we separate the elements of K into n/r disjoint subsets. Yet each subset is a coset of $\langle x \rangle$, so we have a union of left cosets, contradicting our given information. Therefore, the intersection is just the identity.

In the other case, we find a similar result; suppose again that x is in the intersection. Now we know that $|x| = r < \infty$ for some r because our group is a torsion group. Thus for each element in K, we will be able to iterate it and get an $r-$cycle, which is a left coset in G; we again find that we have a union of left cosets, a contradiction.

And we are done.